HOMEWORK 7 (DUE OCTOBER 19)

1. Verify that a short exact sequence of $\mathfrak{g}$-modules

$$0 \longrightarrow V \longrightarrow U \longrightarrow W \longrightarrow 0$$

gives rise to an exact sequence $H^1(\mathfrak{g}, V) \to H^1(\mathfrak{g}, U) \to H^1(\mathfrak{g}, W)$ of vector spaces.

2. Show that an abstract Jordan decomposition in a Lie algebra $\mathfrak{g}$ is unique when it exists if and only if the center of $\mathfrak{g}$ is trivial (that is, $Z(\mathfrak{g}) = 0$).

3. (a) Classify all 2-dimensional Lie algebras over any field $\mathbf{k}$.

(b) Classify all finite-dimensional Lie algebras $\mathfrak{g}$ for which $\dim Z(\mathfrak{g}) = \dim \mathfrak{g} - 2$.

Hint: If $\dim \mathfrak{g} = n$ *show that either* $\mathfrak{g} \simeq \mathrm{Ab}_{n-3} \oplus \mathrm{Heis}_3$ *or* $\mathfrak{g} \simeq \mathrm{Ab}_{n-2} \oplus \mathfrak{a}$*, where* Ab_m *denotes the abelian Lie algebra of dimension* m*,* $\mathfrak{a}$ *is the non-abelian 2-dimensional Lie algebra from part (a), and* Heis_3 *is the Heisenberg algebra from [Homework 5, Problem 5(a)].*

(c) Show that any non-abelian 3-dimensional nilpotent Lie algebra is isomorphic to the Heisenberg algebra Heis_3.

4. Consider the Heisenberg algebra Heis_3 and its representation on $\mathbf{k}[t]$ given by

$$x(f(t)) = f'(t)\,, \quad y(f(t)) = tf(t)\,, \quad z(f(t)) = f(t) \qquad \forall\, f(t) \in \mathbf{k}[t]\,.$$

If $\mathrm{char}(\mathbf{k}) = p > 0$, show that $t^p\mathbf{k}[t]$ is a Heis_3-submodule of $\mathbf{k}[t]$. Verify that while Heis_3 is a solvable Lie algebra, its action on the quotient module $V = \mathbf{k}[t]/t^p\mathbf{k}[t]$ <u>cannot</u> be represented by upper triangular matrices in any basis.

5. Let $\mathbf{k}$ be an algebraically closed field of characteristic $\mathrm{char}(\mathbf{k}) = 2$, and consider the action $\rho\colon \mathrm{Heis}_3 \curvearrowright V = \mathbf{k}[t]/t^2\mathbf{k}[t]$ from the problem above. Verify that $V = V_\lambda$ for some function $\lambda\colon \mathrm{Heis}_3 \to \mathbf{k}$, where V_λ is the common generalized eigenspace for Heis_3 with the eigenvalue λ:

$$V_\lambda = \left\{ v \in V \mid (\rho(x) - \lambda(x))^N v = 0 \;\forall\, N \gg 1 \text{ and all } x \in \mathrm{Heis}_3 \right\}.$$

Compute λ explicitly and show that λ is <u>not</u> a linear function on Heis_3.

6. (a) Show that all derivations of the 2-dimensional non-abelian Lie algebra are inner.

(b) Find all derivations of the Heisenberg algebra Heis_3. Show that not all of them are inner.