# HOMEWORK 8 (DUE OCTOBER 26)

1. Let $\mathfrak{g}$ be a semisimple Lie algebra with a non-degenerate invariant symmetric bilinear form $(\cdot,\cdot)$, and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. For any root $\alpha$ of $\mathfrak{g}$, verify that the element

$$h_\alpha = \frac{2}{(\alpha,\alpha)} H_\alpha \in \mathfrak{h}$$

is independent of the pairing ([Lecture 16, Lemma 2(c)]).

2. Let $\mathfrak{g}$ be a semisimple Lie algebra with a non-degenerate invariant symmetric bilinear form $(\cdot,\cdot)$, and $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Let $R \subset \mathfrak{h}^*$ be the root system of $\mathfrak{g}$. Prove the following **string property** for any $\alpha, \beta \in R$:

$$\big\{\beta + n\alpha \,|\, n \in \mathbb{Z}\big\} \cap \big(R \cup \{0\}\big) = \big\{\beta - p\alpha, \beta - (p-1)\alpha, \ldots, \beta + (q-1)\alpha, \beta + q\alpha\big\}$$

for some $p, q \in \mathbb{Z}_{\geq 0}$ satisfying $p - q = \frac{2(\alpha,\beta)}{(\alpha,\alpha)}$.

3. (a) Verify that $x \in \mathfrak{gl}_n(\mathbb{C})$ is strongly regular iff its eigenvalues are distinct.

(b*) Find a criteria for $x \in \mathfrak{gl}_n(\mathbb{C})$ to be regular in terms of its Jordan normal form.

4. (a) Let $\mathfrak{h} \subset \mathfrak{g}$ be a subspace of diagonal matrices in $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$. Verify that $\mathfrak{h}$ is a Cartan subalgebra. Describe explicitly the root system $R \subset \mathfrak{h}^*$ and the corresponding root subspaces.

(b) Let $\mathfrak{g} = \mathfrak{sp}_{2n}(\mathbb{C})$ be the symplectic Lie algebra realized as

$$\mathfrak{g} = \big\{A \in \mathrm{Mat}_{2n\times 2n}(\mathbb{C}) \,|\, AJ + JA^t = 0\big\} \quad \text{with} \quad J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix},$$

where $I_n \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ is the identity matrix. Let $\mathfrak{h} \subset \mathfrak{g}$ be a subspace of diagonal matrices. Verify that $\mathfrak{h}$ is a Cartan subalgebra. Describe explicitly the root system $R \subset \mathfrak{h}^*$ and the corresponding root subspaces of $\mathfrak{g}$.

(c) Let $\mathfrak{g} = \mathfrak{so}_{2n}(\mathbb{C})$ be the orthogonal Lie algebra realized as

$$\mathfrak{g} = \big\{A \in \mathrm{Mat}_{2n\times 2n}(\mathbb{C}) \,|\, AJ + JA^t = 0\big\} \quad \text{with} \quad J = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}.$$

Let $\mathfrak{h} \subset \mathfrak{g}$ be a subspace of diagonal matrices. Verify that $\mathfrak{h}$ is a Cartan subalgebra. Describe explicitly the root system $R \subset \mathfrak{h}^*$ and the corresponding root subspaces of $\mathfrak{g}$.

(d) Let $\mathfrak{g} = \mathfrak{so}_{2n+1}(\mathbb{C})$ be the orthogonal Lie algebra realized as

$$\mathfrak{g} = \big\{A \in \mathrm{Mat}_{(2n+1)\times(2n+1)}(\mathbb{C}) \,|\, AJ + JA^t = 0\big\} \quad \text{with} \quad J = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & 0 & I_n \\ 0 & I_n & 0 \end{pmatrix}.$$

Let $\mathfrak{h} \subset \mathfrak{g}$ be a subspace of diagonal matrices. Verify that $\mathfrak{h}$ is a Cartan subalgebra. Describe explicitly the root system $R \subset \mathfrak{h}^*$ and the corresponding root subspaces of $\mathfrak{g}$.

5. Let $V$ be a 4-dimensional complex vector space with a basis $e_1, e_2, e_3, e_4$.

(a) Define a bilinear form $B$ on $W = \Lambda^2 V$ via $w_1 \wedge w_2 = B(w_1, w_2) e_1 \wedge e_2 \wedge e_3 \wedge e_4$. Verify that $B$ is a symmetric non-degenerate form on $W$ and construct an orthonormal basis for $B$.

(b) Let $\mathfrak{g} = \{x \in \mathfrak{gl}(W) \mid B(xw_1, w_2) + B(w_1, xw_2) = 0 \ \forall w_1, w_2 \in W\}$. Show that $\mathfrak{g} \simeq \mathfrak{so}_6(\mathbb{C})$.

(c) Show that the form $B$ is invariant under the natural action of $\mathfrak{sl}_4(\mathbb{C})$ on $W$.

(d) Using the above results, construct a Lie algebra isomorphism $\mathfrak{sl}_4(\mathbb{C}) \xrightarrow{\sim} \mathfrak{so}_6(\mathbb{C})$.

6. Let $\mathfrak{g}$ be a complex finite-dimensional Lie algebra which has a decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$$

where $R$ is a finite subset of $\mathfrak{h}^* \setminus \{0\}$, $\mathfrak{h}$ is an abelian Lie subalgebra, and we have

$$[h, x] = \alpha(h) x \qquad \text{for any} \quad h \in \mathfrak{h}, x \in \mathfrak{g}_\alpha, \alpha \in R\,.$$

Assume further that for any $\alpha \in R \cup \{0\}$, the Killing form of $\mathfrak{g}$ gives rise to a

$$\text{non-degenerate pairing} \quad \mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha} \longrightarrow \mathbb{C}\,.$$

Show that $\mathfrak{g}$ is semisimple and $\mathfrak{h}$ is its Cartan subalgebra.