HOMEWORK 9 (DUE NOVEMBER 2)

1. Let $\{e_i\}_{i=1}^n$ be the standard orthonormal basis of $\mathbb{R}^n$. Verify that

$$R := \big\{ \pm e_i, \pm 2e_i \,|\, 1 \leq i \leq n \big\} \cup \big\{ \pm e_i \pm e_j \,|\, 1 \leq i < j \leq n \big\}$$

is a non-reduced root system in $\mathbb{R}^n$ (the so-called $\mathbf{BC}_n$**-type** root system).

2. (a) Let $R \subset E$ be a root system. Verify that the set

$$R^\vee := \big\{ \alpha^\vee \,|\, \alpha \in R \big\} \subset E^*$$

is also a root system (called the **dual root system** of $R \subset E$), where the **coroot** $\alpha^\vee \in E^*$ is defined via

$$\alpha^\vee(x) = \frac{2(\alpha, x)}{(\alpha, \alpha)} \qquad \forall\, x \in E\,.$$

(b) Verify that the root systems $(R^\vee)^\vee \subset (E^*)^*$ and $R \subset E$ are isomorphic.

(c) Let $\Pi = \{\alpha_1, \ldots, \alpha_n\} \subset R$ be the set of simple roots of a reduced root system R with respect to a polarization given by $t \in E^*$. Verify that $\Pi^\vee := \{\alpha_1^\vee, \ldots, \alpha_n^\vee\}$ is the set of simple roots of $R^\vee$ with respect to the polarization given by $t^\vee = \pi^{-1}(t)$, where $\pi\colon E \to E^*$ is the isomorphism induced by the inner product.

3. (a) Given a collection of vectors $\{v_i\}_{i=1}^n$ in a Euclidean space E which satisfy:

- $(v_i, v_j) \leq 0$ for any $i \neq j$
- $t(v_i) > 0$ for some $t \in E^*$ and all $1 \leq i \leq n$

prove that the vectors $\{v_i\}_{i=1}^n$ are linearly independent.

Hint: assuming we have a nontrivial linear combination of v_i's equal to 0, rearrange the terms so that $\sum_{i \in A} c_i v_i = \sum_{i \in B} c_i v_i$ for some disjoint subsets $A, B \subset \{1, \ldots, n\}$ with $c_i > 0$. Show that $A, B \neq \emptyset$. Then compute the inner product of the left-hand and right-hand sides.

(b) Use (a) to show that the set of simple roots Π in a root system $R \subset E$ is a basis of E ([Lecture 19, Theorem 2]).

4. Let $\mathfrak{h}$ be a Cartan subalgebra in a complex semisimple Lie algebra $\mathfrak{g}$. This naturally gives rise to a reduced root system $R \subset \mathfrak{h}^*_{\mathbb{R}}$ (see Propositions 1–2 of Lecture 19).

(a) For $\mathfrak{g} = \mathfrak{sp}_{2n}(\mathbb{C})$ one gets the so-called $\mathbf{C}_n$**-type** root system [Homework 8, Problem 4(b)]

$$R_{C_n} = \big\{ \pm 2e_i \,|\, 1 \leq i \leq n \big\} \cup \big\{ \pm e_i \pm e_j \,|\, 1 \leq i < j \leq n \big\} \subset \mathbb{R}^n\,.$$

Describe the simple roots $\Pi \subset R_{C_n}$ for the polarization given by $t \in (\mathbb{R}^n)^*$ such that

$$t(e_1) \gg t(e_2) \gg \cdots \gg t(e_n) > 0\,.$$

(b) For $\mathfrak{g} = \mathfrak{so}_{2n}(\mathbb{C})$ one gets the so-called $\mathbf{D}_n$**-type** root system [Homework 8, Problem 4(c)]

$$R_{D_n} = \big\{ \pm e_i \pm e_j \,|\, 1 \leq i < j \leq n \big\} \subset \mathbb{R}^n\,.$$

Describe the simple roots $\Pi \subset R_{D_n}$ for the polarization given by $t \in (\mathbb{R}^n)^*$ as in (a).

(c) For $\mathfrak{g} = \mathfrak{so}_{2n+1}(\mathbb{C})$ one gets the **$\mathbf{B}_n$-type** root system [Homework 8, Problem 4(d)]

$$R_{B_n} = \big\{ \pm e_i \,|\, 1 \leq i \leq n \big\} \cup \big\{ \pm e_i \pm e_j \,|\, 1 \leq i < j \leq n \big\} \subset \mathbb{R}^n\,.$$

Describe the simple roots $\Pi \subset R_{B_n}$ for the polarization given by $t \in (\mathbb{R}^n)^*$ as in (a).

5. (a) The F_4 root system $R_{F_4} \subset \mathbb{R}^4$ is defined as the union of the B_4 root system R_{B_4} and

$$\pm\frac{1}{2}e_1 \pm \frac{1}{2}e_2 \pm \frac{1}{2}e_3 \pm \frac{1}{2}e_4 \in \mathbb{R}^4$$

for all 16 choices of signs. Verify that it is a reduced root system. Describe the simple roots $\Pi \subset R_{F_4}$ for the polarization given by $t \in (\mathbb{R}^4)^*$ with $t(e_1) \gg t(e_2) \gg t(e_3) \gg t(e_4) > 0$.

(b) The E_8 root system $R_{E_8} \subset \mathbb{R}^8$ is the union of the root system R_{D_8} and

$$\frac{1}{2}\Big(\underbrace{\pm e_1 \pm e_2 \pm \cdots \pm e_8}_{\text{even number of } - \text{ signs}} \Big) \in \mathbb{R}^8$$

(there are 128 of such vectors). Verify that it is a reduced root system. Describe the simple roots $\Pi \subset R_{E_8}$ for the polarization given by $t \in (\mathbb{R}^8)^*$ with $t(e_1) \gg \cdots \gg t(e_8) > 0$.

(c) The E_7 root system R_{E_7} is defined as the intersection of R_{E_8} with the hyperplane $\{(x_1, \ldots, x_8) \in \mathbb{R}^8 \,|\, x_1 + x_2 = 0\}$ in $\mathbb{R}^8$. Verify that it is a reduced root system. Describe the simple roots $\Pi \subset R_{E_7}$ w.r.t. the polarization induced from (b).

Hint: The simple roots for R_{E_7} form a subset of simple roots for R_{E_8}.

(d) The E_6 root system R_{E_6} is defined as the intersection of R_{E_8} with the codimension two subspace $\{(x_1, \ldots, x_8) \in \mathbb{R}^8 \,|\, x_1 + x_2 = 0, x_2 - x_3 = 0\}$ in $\mathbb{R}^8$. Verify that it is a reduced root system. Describe the simple roots $\Pi \subset R_{E_6}$ w.r.t. the polarization induced from (b).

Hint: The simple roots for R_{E_6} form a subset of simple roots for R_{E_7}.

As you will learn next week, any irreducible reduced root system is isomorphic to one of:

$$A_n\,(n \geq 1)\,,\quad B_n\,(n \geq 2)\,,\quad C_n\,(n \geq 3)\,,\quad D_n\,(n \geq 4)\,,\quad E_6\,,\quad E_7\,,\quad E_8\,,\quad F_4\,,\quad G_2\,.$$

This also provides a complete classification of complex simple Lie algebras.