HOMEWORK 10 (DUE NOVEMBER 9)

1. (a) Show that the assignment of a positive Weyl chamber to any polarization of a root system R provides a bijection between all polarizations $R = R_+ \cup R_-$ and all Weyl chambers.

(b) Show that if two Weyl chambers C, C' are separated by a hyperplane L_α, then $s_\alpha(C) = C'$.

2. Let $Q \subset P$ be the root and weight lattices of an abstract rank r root system $R \subset E$. Pick a polarization $R = R_+ \cup R_-$ and let $\Pi = \{\alpha_1, \ldots, \alpha_r\} \subset R_+$ denote the set of simple roots.

(a) Show that the index $|P/Q|$ equals the determinant of the Cartan matrix $A = \left(\alpha_i^\vee(\alpha_j)\right)_{i,j=1}^r$.

(b) Compute the index $|P/Q|$ for the root systems of types A_r, B_r, C_r, D_r.

(c) Describe the quotient group P/Q explicitly for the root systems of types A_r, B_r, C_r, D_r.

3. (a) Let $w = s_{i_1} \cdots s_{i_\ell}$ be a reduced expression of $w \in W$. Show that then

$$\{\alpha \in R_+ \,|\, w(\alpha) \in R_-\} = \{\beta_1, \ldots, \beta_\ell\} \qquad \text{with} \qquad \beta_k = s_{i_\ell} \cdots s_{i_{k+1}}(\alpha_{i_k})$$

and

$$\{\alpha \in R_+ \,|\, w^{-1}(\alpha) \in R_-\} = \{\widetilde{\beta}_1, \ldots, \widetilde{\beta}_\ell\} \qquad \text{with} \qquad \widetilde{\beta}_k = s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k})\,.$$

(b) Suppose $s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}) \in R_-$. Show that $s_{i_1} \cdots s_{i_{k-1}} s_{i_k}$ is not a reduced expression.

4. Let w_0 be the longest element of the Weyl group W. Show that for any $w \in W$, we have:

$$\ell(ww_0) = \ell(w_0 w) = \ell(w_0) - \ell(w)\,.$$

5. (a) Verify that in the Weyl group $W \simeq S_n$ of the A_{n-1} type root system the longest element w_0 is the permutation $w_0 = (n\ n-1\ \ldots\ 2\ 1)$.

(b) Describe explicitly the Weyl groups W for the root systems of types B_r, C_r, D_r.

(c) Describe explicitly the longest element $w_0 \in W$ for the root systems from (b).

6. Given positive roots $\alpha, \beta, \alpha', \beta' \in R_+$ such that $\alpha + \beta = \alpha' + \beta'$, verify that then

$$\alpha' = \alpha + \gamma \ \text{ and } \ \beta' = \beta - \gamma \qquad \text{or} \qquad \alpha' = \beta + \gamma \ \text{ and } \ \beta' = \alpha - \gamma$$

for some $\gamma \in R \cup \{0\}$.

7. Let $w_0 = s_{i_1} s_{i_2} \cdots s_{i_N}$ (with $N = |R_+|$) be a reduced decomposition of the longest element $w_0 \in W$. By Problem 3 above, we have $R_+ = \{\beta_1, \beta_2, \ldots, \beta_N\}$ with $\beta_k = s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k})$. This gives a natural order on R_+ via: $\beta_i < \beta_j$ iff $i < j$. Show that this order is **convex**:

$$\forall\, \alpha < \beta \in R_+ \quad \text{such that} \quad \alpha + \beta \in R_+ \quad \text{we have} \quad \alpha < \alpha + \beta < \beta\,.$$

In fact, any convex order on R_+ arises via a unique reduced decomposition of w_0 in this way:

$$\{\text{reduced decompositions of } w_0 \in W\} \overset{\text{bijection}}{\longleftrightarrow} \{\text{convex orders on } R_+\}$$