HOMEWORK 11 (DUE NOVEMBER 16)

1. (a) Verify that if two vertices of a Dynkin diagram are connected by a single edge, then the corresponding simple roots are in the same W-orbit.

(b) Show that for a reduced irreducible root system R, the Weyl group W acts transitively on the set of all roots of the same length.

2. Given any irreducible root system $R \subset E$, verify that E is an irreducible representation of the corresponding Weyl group W.

3. (a) Verify that the *classical* reduced root systems of types A_n, B_n, C_n, D_n (from Problem 4 of Homework 9) indeed have the same named Dynkin diagrams (as drawn in Lecture 22).

(b) Verify that the *exceptional* reduced root systems of types E_6, E_7, E_8, F_4, G_2 (see Problem 5 of Homework 9) indeed have the same named Dynkin diagrams (as drawn in Lecture 22).

4. Given a reduced root system $R \subset E$ with a polarization $R = R_+ \cup R_-$ and the set of simple roots $\Pi = \{\alpha_1, \ldots, \alpha_r\} \subset R_+$, recall the element $\rho = \frac{1}{2}\sum_{\alpha\in R_+} \alpha = \sum_{i=1}^{r} \varpi_i \in E$ (see Lemma 2 of Lecture 21). Let $\rho^\vee \in E^*$ be such an element for the dual root system $R^\vee \subset E^*$.

(a) Compute $\rho, \rho^\vee$ for the *classical* root systems of types A_n, B_n, C_n, D_n.

(b*) Compute $\rho, \rho^\vee$ for the *exceptional* root systems of types E_6, E_7, E_8, F_4, G_2.

5. Complete the proof of the Main Theorem from Lecture 22 by verifying that none of the following graphs can appear as a subgraph of a Dynkin diagram (see pictures in the notes):

- analogue of $\widetilde{D}_n$ with some multiple edges
- $\widetilde{G}_2$, $D_4^{(3)}$ as well as their analogues with a multiple edge instead of the simple one
- $\widetilde{B}_n$, $A_{2n-1}^{(2)}$ as well as their analogues with one/two of their two "legs" being multiple
- $\widetilde{E}_6$, $\widetilde{E}_7$, $\widetilde{E}_8$
- $\widetilde{C}_n, D_{n+1}^{(2)}, A_{2n}^{(2)}$
- $\widetilde{F}_4, E_6^{(2)}$

Hint: Verify that in each of the above cases the corresponding Cartan matrix is degenerate, or alternatively, verify that the symmetrized Cartan matrix is not positive definite.

6. Let G be a connected $\mathbb{C}$ Lie group such that $\mathfrak{g} = \mathrm{Lie}(G)$ is semisimple. Fix a root decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in R} \mathfrak{g}_\alpha$. For any $\alpha \in R$, consider the embedding $\iota_\alpha \colon \mathfrak{sl}(2,\mathbb{C}) \hookrightarrow \mathfrak{g}$ [Lecture 16, Lemma 2], and lift it to $\iota_\alpha \colon SL(2,\mathbb{C}) \to G$. Define $S_\alpha := \iota_\alpha(\exp(f_\alpha)\exp(-e_\alpha)\exp(f_\alpha)) \in G$. Show that the dual of the adjoint action of S_α on $\mathfrak{g}^*$ preserves $\mathfrak{h}^*$, and $\mathrm{Ad}^*(S_\alpha)|_{\mathfrak{h}^*}$ coincides with the reflection $s_\alpha \colon \mathfrak{h}^* \to \mathfrak{h}^*$. Deduce that the Weyl group W acts on $\mathfrak{h}^*$ by inner automorphisms, i.e. for any $w \in W$ there is a (non-unique!) element $\tilde{w} \in G$ such that $\mathrm{Ad}^*(\tilde{w})|_{\mathfrak{h}^*} = w$.