HOMEWORK 12 (DUE NOVEMBER 28)

1. This problem is aimed at completing the details in the proof of [Lecture 23, Theorem 2]. Let R be a reduced root system with a Cartan matrix $A = (a_{ij})_{i,j=1}^r$.

(a) Let $\widetilde{\mathfrak{g}}(R)$ be a Lie algebra generated by $\{e_i, h_i, f_i\}_{i=1}^r$ subject to Chevalley relations:

$$[h_i, h_j] = 0\,, \quad [h_i, e_j] = a_{ij} e_j\,, \quad [h_i, f_j] = -a_{ij} f_j\,, \quad [e_i, f_j] = \delta_{ij} h_i \qquad \forall\, i, j\,.$$

Let $\widetilde{\mathfrak{n}}_+(R)$ be the subalgebra generated by $\{e_i\}_{i=1}^r$, $\widetilde{\mathfrak{n}}_-(R)$ be the subalgebra generated by $\{f_i\}_{i=1}^r$, $\widetilde{\mathfrak{h}}_+(R)$ be the subalgebra generated by $\{h_i\}_{i=1}^r$. Verify that as vector spaces:

$$\widetilde{\mathfrak{g}}(R) = \widetilde{\mathfrak{n}}_-(R) \oplus \widetilde{\mathfrak{h}}(R) \oplus \widetilde{\mathfrak{n}}_+(R)\,.$$

(b) Prove that $\widetilde{\mathfrak{n}}_+(R)$ is a free Lie algebra in $\{e_i\}_{i=1}^r$, $\widetilde{\mathfrak{n}}_-(R)$ is a free Lie algebra in $\{f_i\}_{i=1}^r$, and $\widetilde{\mathfrak{h}}_+(R)$ has a basis $\{h_i\}_{i=1}^r$.

Hint: Consider a Lie algebra $\mathfrak{a} = \bar{\mathfrak{h}} \ltimes L_r$, *where* $\bar{\mathfrak{h}}$ *is an abelian Lie algebra with a basis* $\{\bar{h}_i\}_{i=1}^r$, L_r *is a free Lie algebra generated by* $\{\bar{f}_i\}_{i=1}^r$, *and the semidirect product is with respect to* $[\bar{h}_i, \bar{f}_j] = -a_{ij}\bar{f}_j$ *([Homework 5, Problem 7]). Consider the universal enveloping algebra*

$$U(\mathfrak{a}) = \mathbb{C}[\bar{h}_1, \ldots, \bar{h}_r] \ltimes \mathbb{C}\langle \bar{f}_1, \ldots, \bar{f}_r \rangle\,.$$

Construct the unique action of $\widetilde{\mathfrak{g}}(R)$ *on* $U(\mathfrak{a})$ *such that*

$$h_i(H \otimes 1) = (\bar{h}_i H) \otimes 1\,, \quad f_i(H \otimes F) = H \otimes (\bar{f}_i F)$$

for any $H \in \mathbb{C}[\bar{h}_1, \ldots, \bar{h}_r]$ *and* $F \in \mathbb{C}\langle \bar{f}_1, \ldots, \bar{f}_r\rangle$. *Consider the linear map*

$$\varphi\colon \widetilde{\mathfrak{g}}(R) \to U \quad \text{given by} \quad x \mapsto x(1)\,.$$

Deduce that $\{h_i\}_{i=1}^r \subset \widetilde{\mathfrak{g}}(R)$ *are linearly independent (since so are* $\varphi(h_i) \in U(\mathfrak{a})$*), hence form a basis of* $\widetilde{\mathfrak{h}}(R)$. *As the restriction of* φ *to* $\widetilde{\mathfrak{n}}_-(R)$ *sends any commutator of* f_i*'s to the corresponding commutator of* $\bar{f}_i$*'s, deduce that* $\widetilde{\mathfrak{n}}_-(R)$ *is indeed a free Lie algebra in* $\{f_i\}_{i=1}^r$.

(c) For any k and $i \neq j$, verify that $[f_k, \mathrm{ad}(e_i)^{1-a_{ij}} e_j] = 0 = [e_k, \mathrm{ad}(f_i)^{1-a_{ij}} f_j]$ in $\widetilde{\mathfrak{g}}(R)$.

(d) Show that if $a, b \in \widetilde{\mathfrak{g}}(R)$ generate finite-dimensional $\mathfrak{sl}(2,\mathbb{C})_{\alpha_i}$-submodules for every i, then so does their Lie bracket $[a,b] \in \widetilde{\mathfrak{g}}(R)$.

2. Let $R \subset E$ be a reduced irreducible simply laced root system. Up to rescaling R, we shall assume that $(\alpha, \alpha) = 2$ for any root $\alpha \in R$. Let $Q \subset E$ be the root lattice of R.

(a) Verify that $R = \{\alpha \in Q \,|\, (\alpha,\alpha) = 2\}$.

(b) Construct $\epsilon\colon Q \times Q \to \{\pm 1\}$ satisfying (for any $\alpha, \beta, \gamma \in Q$)

$$\epsilon(\alpha, \beta+\gamma) = \epsilon(\alpha,\beta)\epsilon(\alpha,\gamma)\,, \quad \epsilon(\alpha+\beta, \gamma) = \epsilon(\alpha,\gamma)\epsilon(\beta,\gamma)\,, \quad \epsilon(\alpha,\alpha) = (-1)^{(\alpha,\alpha)/2}\,.$$

Hint: If $\Pi = \{\alpha_1, \ldots, \alpha_r\}$ *are the simple roots of* R *with respect to some polarization, then set* $\epsilon(\alpha_i, \alpha_i) = -1$, $\epsilon(\alpha_i,\alpha_j)\epsilon(\alpha_j,\alpha_i) = (-1)^{(\alpha_i,\alpha_j)}$ *for* $i \neq j$, *and extend it bilinearly to* $Q \times Q$.

(c*) Consider a vector space V with a basis $\{u_i\}_{i\in I}\cup\{v_\alpha\}_{\alpha\in R}$, where $\{\alpha_i\}_{i\in I}$ are simple roots of R with respect to some polarization. Verify that linear operators $e_i, f_i, h_i\colon V\to V$ given by

$$h_i(u_j)=0\,,\qquad h_i(v_\alpha)=(\alpha,\alpha_i)v_\alpha\,,$$

$$e_i(u_j)=|(\alpha_i,\alpha_j)|v_{\alpha_i}\,,\qquad e_i(v_\alpha)=\begin{cases}v_{\alpha+\alpha_i} & \text{if } \alpha+\alpha_i\in R\\ u_i & \text{if } \alpha=-\alpha_i\\ 0 & \text{if } \alpha+\alpha_i\notin R\cup 0\end{cases}\,,$$

$$f_i(u_j)=|(\alpha_i,\alpha_j)|v_{-\alpha_i}\,,\qquad f_i(v_\alpha)=\begin{cases}v_{\alpha-\alpha_i} & \text{if } \alpha-\alpha_i\in R\\ u_i & \text{if } \alpha=\alpha_i\\ 0 & \text{if } \alpha-\alpha_i\notin R\cup 0\end{cases}$$

give rise an action $\mathfrak{g}\curvearrowright V$. Identify this $\mathfrak{g}$-representation with the adjoint representation of $\mathfrak{g}$.

This reconstructs the simply-laced simple Lie algebras from their Cartan matrices.

3. Let $\mathfrak{h}$ be a Cartan subalgebra of a semisimple Lie algebra $\mathfrak{g}$. Show that any submodule and quotient of a $\mathfrak{g}$-module that admits a weight decomposition also admit such decompositions.

4. Let $\mathfrak{g}$ be a semisimple Lie algebra with a triangular decomposition $\mathfrak{g}=\mathfrak{n}_-\oplus\mathfrak{h}\oplus\mathfrak{n}_+$. Prove that any nonzero $\mathfrak{g}$-module homomorphism $\phi\colon M_\lambda\to M_\mu$ of Verma modules is injective.

Hint: Use $M_\sharp\simeq U(\mathfrak{n}_-)$ ([Lecture 24, Prop. 1]) and $U(\mathfrak{n}_-)$ is a domain ([Lecture 9, Cor. 1]).

5. Let P_+ denote the set of all dominant integral weights of $\mathfrak{g}$ in the weight lattice $P=\oplus_i\mathbb{Z}\omega_i$, and consider the subset Q_+ of the root lattice $Q=\oplus_i\mathbb{Z}\alpha_i$ given by $Q_+=\{\sum_i n_i\alpha_i\,|\,n_i\in\mathbb{Z}_{\geq 0}\}$.

(a) Verify that for any $\lambda\in P$, the intersection $P_+\cap(\lambda-Q_+)$ is a finite set.

(b) Verify that for any $\lambda\in P$, the orbit $W\lambda=\{w(\lambda)\,|\,w\in W\}$ has exactly one element in P_+.

Hint: Assume that $\lambda\neq\mu\in P_+$ and $\mu=w(\lambda)$ with the shortest reduced decomposition $w=s_{i_1}s_{i_2}\ldots s_{i_\ell}$. Find $\gamma\in R_+$ such that $s_{i_1}w(\gamma)\in R_+$, $w(\gamma)\in R_-$. Using $(\lambda,\gamma)=(\mu,w(\gamma))$ deduce that $(\lambda,\gamma)=0$. Conclude that $\mu=s_{i_2}\ldots s_{i_\ell}(\lambda)$, contradicting the mimimality of $\ell(w)$.

6. This problem revises [Homework 4, Problem 6] in the context of Lectures 24–25.

(a) For any $k\geq 0$, consider the representation $V=S^k(\mathbb{C}^n)$ of $\mathfrak{sl}_n(\mathbb{C})$. Compute all weights of V and describe the corresponding weight subspaces. Prove that V is irreducible by showing that the space of highest weight vectors is one-dimensional (describe it explicitly).

(b) For any $1\leq k\leq n$, consider the representation $V=\Lambda^k(\mathbb{C}^n)$ of $\mathfrak{sl}_n(\mathbb{C})$. Compute all weights of V and describe the corresponding weight subspaces. Prove that V is irreducible by showing that the space of highest weight vectors is one-dimensional (describe it explicitly).