Lecture #1

While our main focus of the course will be on the Lie algebras, we shall spend the first few lectures on Lie groups which provide the motivation for the subject. To this end, we shall start by recalling the basics of manifolds, as Lie groups are nothing else than manifolds with compatible group structures.

• As one can introduce similar definitions in various C^k-setups, we start first from:

Def 1: A topological group G is a group that is also a topological space such that both $m: G\times G \to G$ and $i: G \to G$ are continuous
(m — multiplication, i — inverse map)

Recall: Notions of topological spaces, closed subsets, product topology and continuous map ($f: X \to Y$ is continuous if $f^{-1}(U) \subseteq X$ - open $\forall$ open $U\subseteq Y$)

This is a very general notion and we shall be interested in a much more refined/special case.

Def 2: A Hausdorff topological space X is an n-dim. topological manifold if it has a countable base and is locally homeomorphic to $\mathbb{R}^n$.

Recall: a) X - Hausdorff if $\forall x\neq y \in X$ $\exists$ open $x\in U$, $y \in V$ s.t. $U \cap V = \emptyset$
(implies uniqueness of limits of sequences)

b) $\{U_\alpha\}$ - base of topology if each U_α - open and any open set is a union of U_α.

c) X is locally homeomorphic to $\mathbb{R}^n$ if $\forall x\in X$ $\exists$ open $x \in U \subseteq X$ and a continuous map $\varphi: U \to \mathbb{R}^n$ such that $\varphi: U \xrightarrow{\sim} \underbrace{\varphi(U)}_{\text{open in } \mathbb{R}^n}$ is a homeomorphism, i.e. it is a bijection and φ^{-1} is also continuous.

Remark: The number n featuring in Def 2 is uniquely determined and is called the dimension of X.

Basic Example: n-dimensional sphere $S^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \ldots + x_{n+1}^2 = 1\} \subseteq \mathbb{R}^{n+1}$

So: each topological manifold X admits an atlas of local charts, i.e. a collection $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in I}$ with $U_\alpha \subseteq X$ - open, $\varphi_\alpha: U_\alpha \to \mathbb{R}^n$ as above, and $X = \bigcup_{\alpha \in I} U_\alpha$.
In other words, $\{U_\alpha\}_{\alpha\in I}$ - open cover of X with "local coordinates" given by φ_α.

The key information is provided by interaction of charts. To this end, if $(U_\alpha, \varphi_\alpha)$, (U_β, φ_β) are two charts with $U_\alpha \cap U_\beta \neq \emptyset$, then we get

$$\boxed{\text{transition map } \tau_{\alpha\beta} = \varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)} \leftarrow \text{homeomorphism of open subsets of } \mathbb{R}^n$$

Still we shall need a much more restricted notion (as our ultimate goal is to pass to linear approximation provided by Lie algebras). Thus, define:

Def 3: For X as in Def 2, its atlas is called of regularity class C^k if all the transition maps $\tau_{\alpha\beta}$ between its charts are of class C^k (that is, k times continuously differentiable). Likewise, an atlas is called real analytic if all $\tau_{\alpha\beta}$ are real analytic (that is, given by its Taylor series expansion in some nbhd of every point). Finally, for even $n = 2m$ (so that $\mathbb{R}^n = \mathbb{C}^m$) an atlas is called complex analytic if all $\tau_{\alpha\beta}$ are complex analytic (a.k.a. holomorphic).

In what follows, we shall not distinguish b/w compatible atlases, i.e. atlases $\{(U_\alpha, \varphi_\alpha)\}$ and $\{(V_\beta, \psi_\beta)\}$ of the same class C^k/real analytic/cpx. analytic s.t. the transition maps b/w U_α & V_β are in the same class.

Def 4: A C^k/real analytic/complex analytic structure on a topological manifold X is an equivalence class of such atlases. Once X is equipped with such a structure, we call X to be a C^k/real analytic/complex analytic manifold. A diffeomorphism (a.k.a. isomorphism) between such manifolds is a homeomorphism which respects the classes of atlases.

Terminology:
- Complex analytic X is usually called just a complex manifold
- C^∞-structure on X is usually referred to as smooth manifold.

Remark: Given two topological manifolds X, Y of the same class (C^k/real anal/cpx anal) their Cartesia product is naturally a manifold of the same type.

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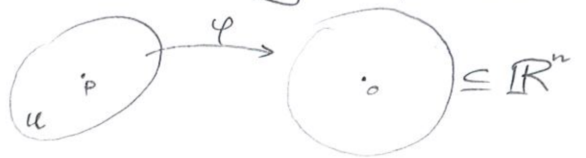
Now we are ready to proceed to key definitions of regular functions & tangent spaces.



consider a local chart (U, φ) s.t. $\varphi(P) = o \in \mathbb{R}^n$ (we can always do parallel shift to achieve this)

Then, composing φ with n projections of $\mathbb{R}^n$ on coordinate axes, we get $\boxed{\text{local coordinates } \; x_1, \dots, x_n : U \to \mathbb{R} \text{ (or } \mathbb{C})}$ which uniquely determine points in U.

Def 5: Given a C^k/real analytic/cpx analytic manifold X, a function $f: V \to \mathbb{R}$ (or $f: V \to \mathbb{C}$ if X - complex analytic) on open $V \subseteq X$ is called *regular function* if $f \circ \varphi_\alpha^{-1} : \varphi_\alpha(V \cap U_\alpha) \to \mathbb{R}$ (or $\mathbb{C}$) is of the same regularity class for some (equivalently, any) atlas $\{(U_\alpha, \varphi_\alpha)\}$ defining the corresponding structure on X.

Notation: $\mathcal{O}(V) = \{$the space of regular f-s on $V\}$

Given two open nbhds U, V of the point $P \in X$ and two regular f-s $f \in \mathcal{O}(U)$, $g \in \mathcal{O}(V)$ we shall say they are *equal near P* if $\exists$ open $P \in W \subseteq U \cap V$ s.t. $f|_W = g|_W \in \mathcal{O}(W)$. This gives rise to important notion of:

Def 6: A *germ* of a regular function at P is the corresponding equivalence class of regular functions defined on some nbhd of P equal near P.

Notation: $\mathcal{O}_P = \{$the algebra of germs of regular f-s at $P\}$
$= \varinjlim \mathcal{O}(U)$
taken over nbhds U of P w.r.t. inclusion/subset relation.

With the above notion in mind, we define:

Def 7: A *derivation at P* is a linear map $D: \mathcal{O}_P \to \mathbb{R}$ (or $\mathbb{C}$) satisfying Leibniz rule:

$$\boxed{D(f \cdot g) = D(f) \cdot g(P) + D(g) \cdot f(P)}$$

Def 8: The space of all such derivations is denoted by $T_P X$ and is called the *tangent space to X at point P*, while elements of $T_P X$ are called *tangent vectors to X at P*.

In what follows, we shall work only with C^∞/real analytic/smooth analytic case

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As every $f \in \mathcal{O}(\mathcal{U})$ for some nbhd $P \in \mathcal{U}$ defines an elt of $\mathcal{O}_P$, we can define $\partial_v(f) \in \mathbb{R}$ (or $\mathbb{C}$) giving rise to $\partial_v : \mathcal{O}(\mathcal{U}) \to \mathbb{R}$ (or $\mathbb{C}$).

Lemma 1: If $x_1, \ldots, x_n$ are local coordinates at P, then $T_P X$ is n-dim with a basis $D_1, \ldots, D_n$ defined by $D_i(f) = \frac{\partial f}{\partial x_i}(0)$.

▶ As $\det(D_i(x_j)) \neq 0$, the elements $D_1, \ldots, D_n$ are lin. indep. To prove they form a basis of $T_P X$, we can assume $P = 0 \in \mathbb{R}^n$ (or $\mathbb{C}^n$) $= X$ and take some $D \in T_P X$. We claim that $\hat{D} := D - \sum_{i=1}^{n} D(x_i) \cdot D_i$ is zero. Indeed, we have $\hat{D}(x_i) = 0$ for all $1 \le i \le n$, and by using Leibniz rule it thus suffices to show that any $f \in \mathcal{O}(\mathcal{U})$ (↑small nbhd of 0) with $f(0) = 0$, one can write it (non-uniquely):

$$\boxed{f(x_1, \ldots, x_n) = x_1 \cdot g_1(x_1, \ldots, x_n) + \ldots + x_n \cdot g_n(x_1, \ldots, x_n)}$$

(as then $\hat{D} f = \sum_{i=1}^{n} x_i|_0 \cdot \hat{D}(g_i) + \sum_{i=1}^{n} \hat{D}(x_i) \cdot g_i|_0 = 0$)

But the above f-la follows immediately from the fundamental theorem:

$$f(x_1, \ldots, x_n) - f(0, \ldots, 0) = \int_0^1 \frac{\partial}{\partial t} f(tx_1, \ldots, tx_n)\, dt = \sum_{i=1}^{n} x_i \cdot \underbrace{g_i(x_1, \ldots, x_n)}_{= \int_0^1 (D_i f)(tx_1, \ldots, tx_n) dt}$$

■

With tangent spaces defined above, we can now introduce differential of regular maps.

Def 9: A continuous map $F : X \to Y$ b/w manifolds of the same class is called *regular* if it's expressed by regular functions in local coordinates (i.e. $\forall h \in \mathcal{O}(\mathcal{U})$, the composition $h \circ F \in \mathcal{O}(F^{-1}(\mathcal{U}))$).

This allows to consider the appropriate category (as composition of regular maps is regular).

Def 10: Given a regular map $F : X \to Y$ and a point $P \in X$, we define the *differential of F at P*, denoted $d_P F$, as the linear map

$$\boxed{T_P X \to T_{F(P)} Y \text{ defined by } (d_P F \cdot v)(f) \overset{\text{def}}{=} v(f \circ F) \quad \forall v \in T_P X,\ f \in \mathcal{O}_{F(P)}}$$

Notation: A shorter notation is just $F_* v$.

Trivial Exercise: Verify the chain rule (and spell it explicitly).

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Def 11: A regular map of manifolds $F: X \to Y$ s.t. $d_P F: T_P X \twoheadrightarrow T_{F(P)} Y$ (surjective) for all $P \in X$ is called a submersion.

As a simple corollary of the implicit function theorem, we have:

Proposition 1: If $F: X \to Y$ is a submersion, then $\forall y \in Y$ its preimage $F^{-1}(y)$ is a manifold of dimension $\dim X - \dim Y$.

Exercise$^{(*)}$: Prove it.

There is also the dual notion:

Def 12: A regular map of manifolds $F: X \to Y$ s.t. $d_P F: T_P X \hookrightarrow T_{F(P)} Y$ (injective) for all $P \in X$ is called an immersion.

Def 13: An immersion $F: X \to Y$ is called an embedding if the map $F: X \to \underbrace{F(X)}_{\text{with induced from } Y \text{ topology}}$ is a homeomorphism. We also say then that $F(X) \subseteq Y$ is an embedded submanifold.

Warning: In general immersion $\not\Leftrightarrow$ injective

Exercise: Provide counterexamples for both directions, i.e.

- an injective map which is not an immersion
- an immersion which is not injective.

Def 14: An embedding $F: X \to Y$ of manifolds (see Def 13) is called closed if $F(X) \subseteq Y$ is a closed subset. We also say then that $F(X) \subseteq Y$ is a closed embedded submanifold of Y.

Exercise$^{(*)}$: Show that if $F: X \to Y$ is an immersion, then $\forall x \in X$, there are local coordinates in a nbhd of $x \in X$ and nbhd of $F(x) \in Y$ such that F is given just by

$$(x_1, \dots, x_n) \longmapsto (x_1, \dots, x_n, 0, \dots, 0)$$