Lecture #2

Today we shall discuss the key notion of Lie groups, which shall give rise to Lie algebras next week.

Def 1: A C^k $(k=0,1,\dots,\infty)$ / real analytic / complex analytic **Lie group** is a manifold G of the same class, with a group structure s.t. the multiplication $m: G\times G \to G$ is regular.

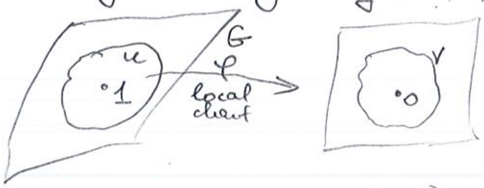
Note that we actually do not need to require that the inversion map $\imath: G\to G$ is regular, since it will actually follow from Def 1. But before we prove it, let's note that the left and right multiplications are diffeomorphisms $\forall g$:

$$\boxed{L_g: G \to G,\ x \mapsto gx \qquad \text{and} \qquad R_g: G \to G,\ x \mapsto x\cdot g}$$

Remark: In fact, any C^0 Lie group is actually a real analytic group (this was one of Hilbert's 20 problems, solved in the 50s). For that reason, we shall always consider only real/complex analytic Lie gps.

Lemma 1: In a Lie group (see Def 1), the inverse map $\imath: G \to G$, $g \mapsto g^{-1}$ is a diffeomorphism. Furthermore, its differential $d_1\imath$ at the unit elt 1 of G is $-\mathrm{Id}$ (minus identity).

▶ First, we note that it suffices to check regularity of $\imath$ in the nbhd of $1\in G$ (indeed, if $\mathcal{U}$ - open nbhd of 1, then $g\cdot\mathcal{U}$ - open nbhd of $g\in G$ $\forall g$, and $(gx)^{-1} = x^{-1}\cdot g^{-1} = R_{g^{-1}}(x^{-1})$, so that the problem can be easily reduced to nbhd of 1)



$\mathbb{R}^n \rightsquigarrow$ Local coordinates $x_1,\dots,x_n$ on a nbhd $\mathcal{U}$ of $1\in G$

Let's see what the multiplication map looks like in $\{x_i\}$.

As $m(1,1)=1$, $m(g,1)=g=m(1,g)$, we conclude that

$$m(\vec{0},\vec{0})=\vec{0},\quad m(\vec{x},\vec{0})=\vec{x},\quad m(\vec{0},\vec{x})=\vec{x} \quad \left(\Rightarrow m(\vec{x},\vec{y}) \sim \vec{x}+\vec{y} \text{ near } \vec{0}\right)$$

We can thus think of m as a map $\mathbb{R}^n\times\mathbb{R}^n \supseteq \underset{\text{of } \vec{0}}{\text{open nbhd}} \longrightarrow \mathbb{R}^n$.

As the Jacobian of this map w.r.t. 2^{nd} copy of $\mathbb{R}^n$ is $I_n = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1\end{pmatrix}_{n\times n}$ at point $\vec{0}$

(to this end, write m in coordinates as $(m_1,\dots,m_n)$, and consider the $n\times n$ matrix $\left(\frac{\partial m_i}{\partial y_j}\right)_{i,j=1}^n$, $(y_1,\dots,y_n)$ - coordinates of 2^{nd} copy $\mathbb{R}^n$)

we can apply the implicit function theorem to deduce that $m(\vec{x},\vec{y})=\vec{0}$ is solved by a regular function $\vec{y}=\imath(\vec{x})$ in the nbhd of $\vec{0}$. Also implicit f-n thm implies that $d_0\imath = -\mathrm{Id}$

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Before proceeding to examples, let us note that besides for defining objects of interests (Lie groups) we also need to define morphisms b/w them:

Def 2: a) A homomorphism of Lie groups $f: G \to H$ is a group homomorphism that is also a regular map.

b) A homomorphism f is called isomorphism if f is a group isomorphism and f^{-1} - regular.

Basic Examples

1) $(\mathbb{R}^n, +)$ - real Lie group

$(\mathbb{C}^n, +)$ - complex Lie group

2) $(\mathbb{R}^*, \cdot)$, $(\mathbb{R}_{>0}, \cdot)$ - real Lie gps

$(\mathbb{C}^*, \cdot)$ - complex Lie group

3) S^1 (unit circle) $= \{z \in \mathbb{C} \mid |z| = 1\}$ - real Lie gp (under multiplication)

4) $GL_n(\mathbb{R}) = \{$invertible $n \times n$ matrices over $\mathbb{R}\}$ - real Lie gp (under multiplication)

$GL_n(\mathbb{C}) = \{$ -"- over $\mathbb{C}\}$ - complex Lie gp (under multiplication)

5) Every finite (or countable) group G with discrete topology is a real and complex Lie group.

6) $\underset{\text{special unitary group}}{SU(2)} = \{X \in GL_2(\mathbb{C}) \mid X \cdot \overline{X}^t = Id, \ \det X = 1\}$ is a 3-dimensional Lie group.

Let's take a closer look at $SU(2)$. Write $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ so that $\overline{X}^t = \begin{pmatrix} \bar a & \bar c \\ \bar b & \bar d \end{pmatrix}$

Then: $X \cdot \overline{X}^t = Id \Leftrightarrow \underbrace{a\bar a + b\bar b = 1}_{|a|^2+|b|^2=1}, \ \underbrace{a\bar c + b\bar d = 0}_{\Updownarrow \ (c,d) = \gamma \cdot (-\bar b, \bar a) \text{ for some } \gamma \in \mathbb{C}}, \ \underbrace{c\bar c + d\bar d = 1}_{|c|^2+|d|^2=1} \Big\} \Rightarrow |\gamma|^2 = 1.$

$\det X = 1 \overset{\text{above}}{\Longleftrightarrow} a\underset{\gamma\bar a}{\underbrace{d}} - b\underset{-\gamma\bar b}{\underbrace{c}} = 1 \Leftrightarrow \gamma(a\bar a + b\bar b) = 1 \Leftrightarrow \gamma = 1.$

SO: $SU(2) = \left\{\begin{pmatrix} a & b \\ -\bar b & \bar a \end{pmatrix} \in GL_2(\mathbb{C}) \,\middle|\, |a|^2 + |b|^2 = 1\right\} \underset{\substack{a = x_1 + ix_2 \\ b = x_3 + ix_4}}{=} \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\} = S^3$

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As the theory of finite (or countable) groups with discrete topology is already nontrivial, it makes sense to separate this part. To this end, let us start by evoking the notions of connected components and path-connected components.

Def 3 : a) A topological space X is connected if the only subsets of X that are both open & closed are just $\emptyset$ and X. For any topological space X and a point $x \in X$, the connected component of x in X is the union of all connected subsets of X which contain point x (i.e. largest connected subset $\ni x$).

b) A topological space X is path-connected if $\forall x, y \in X$ $\exists$ continuous map $\gamma : [0,1] \to X$ s.t. $\gamma(0) = x$, $\gamma(1) = y$.

For any topological space X and a point $x \in X$, the path-connected component of x in X is the set of all $y \in X$ for which there exists a continuous $\gamma : [0,1] \to X$ with $\gamma(0) = x$, $\gamma(1) = y$.

Exercise(*) : a) Show that a path-connected space is connected.

b) Show that $\{(0,y) \mid -1 \le y \le 1\} \cup \{(x, \sin(\frac{1}{x})) \mid 0 < x \le 1\}$ is connected but not path-connected

c) Show that any connected manifold is path-connected.

Due to part c) of the above exercise, the two notions coincide for manifolds.

Notation : a) $\pi_0(X)$ will denote the set of $\overset{\text{path-connected}}{\text{connected}}$ components of X.

b) For a real or complex Lie group G, we shall use G° to denote the connected component of $1 \in G$.

Obvious : $\forall g \in G$ the connected component of g is just $g \cdot G^\circ$

Lemma 2 : G° is a normal subgroup of G

- We just need to check that $g \cdot x \cdot g^{-1} \in G^\circ$ $\forall g \in G, x \in G^\circ$.

But : $x \in G^\circ$ means that $\exists$ continuous $\gamma : [0,1] \to G$ with $\gamma(0) = 1$, $\gamma(1) = x$

Consider $\hat{\gamma} : [0,1] \to G$ defined via $\hat{\gamma}(t) = g \cdot \gamma(t) \cdot g^{-1} = R_{g^{-1}} L_g \, \gamma(t)$

It's clearly continuous (as so are γ, L_g, $R_{g^{-1}}$) and $\hat{\gamma}(0) = 1$, $\hat{\gamma}(1) = gxg^{-1}$

- As $m(1,1) = 1$, $\imath(1) = 1$ and the image of connected topological space under continuous map is connected, we see that G° is indeed a subgroup

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To conclude the above discussion, let's recall the notion of a quotient topology

[if X is a topological space equipped with a surjective map $p: X \twoheadrightarrow Y$ to a set Y, then Y is naturally equipped with a quotient topology where $U \subseteq Y$ is called open if $p^{-1}(U) \subseteq X$ - open.]

Lemma 3: $\pi_0(G) = G/G^\circ$ with quotient topology is a discrete countable group.

► By construction, preimage of any subset of G/G° under $G \twoheadrightarrow G/G^\circ$ is a union of connected components, hence, is open. This shows that G/G° is discrete.
The fact that G/G° is at most countable follows from G having a countable base ■

Thus, we morally can always reduce to the study of connected Lie groups. More so, one can further reduce to the study of simply-connected Lie groups that we discuss next. To do so, we start by evoking some theory of coverings

Def 4: A continuous map $\pi: E \to X$ of (Hausdorff) topological spaces is called a covering if $\forall x \in X$ $\exists$ open nbhd $x \in U \subseteq X$ s.t. $\pi^{-1}(U)$ is a union of disjoint open sets (a.k.a. sheets) each homeomorphic to U under π, i.e.
$\pi^{-1}(U) = \coprod V_\alpha$ and $V_\alpha \xrightarrow{\pi} U$ - homeomorphism $\forall \alpha$.

In other words, π locally looks as a projection $F \times X \to X$ for some discrete F.

Basic Example: a) $S^1 \to S^1$ given by $z \mapsto z^n$ $\forall n \geq 1$

b) $\mathbb{R} \to S^1$ given by $z \mapsto e^{iz}$

As for every pt $p \in E$ $\exists$ open nbhd $p \in U$ s.t. $\pi_{|U}: U \to \pi(U)$ - homeomorphism, we see that π is a regular map, and also if X was a manifold then so is E.

[We also note that if we have two coverings
$$\begin{tikzcd} E \arrow[rr, dashed, "h"] \arrow[dr, "\pi"'] & & E' \arrow[dl, "\pi'"] \\ & X & \end{tikzcd}$$
then they are called equivalent if $\exists$ homeom. h making the diagram commutative.]

Def 5: a) Two paths $\gamma_0, \gamma_1: [0,1] \to X$ s.t. $\gamma_i(0) = x$, $\gamma_i(1) = y$ are called homotopic if there is a continuous map $\gamma: [0,1] \times [0,1] \to X$ s.t.
$\gamma(t,0) = \gamma_0(t)$, $\gamma(t,1) = \gamma_1(t)$, $\gamma(0,s) = x$, $\gamma(1,s) = y$.
Such γ is called a homotopy b/w γ_0 and γ_1

b) A path-connected top. space X is called simply connected if $\forall x, y \in X$ $\forall$ paths $\gamma_0, \gamma_1: [0,1] \to X$ b/w x, y are homotopic.

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With the previous definition in mind, we can now state an important property of coverings (see Def 4)

Homotopy Lifting Property of coverings:

(1) $\forall x \in X$ $\forall \tilde{x} \in \pi^{-1}(x) \in E$ and any path $\gamma : [0,1] \to X$ with $\gamma(0) = x$
$\exists!$ path $\tilde{\gamma} : [0,1] \longrightarrow E$ s.t. $\tilde{\gamma}(0) = \tilde{x}$ and $\pi(\tilde{\gamma}(t)) = \gamma(t)$.

(2) If γ_1, γ_2 - homotopic paths in $X \Rightarrow \underbrace{\tilde{\gamma}_1, \tilde{\gamma}_2}_{\text{lifts from (1)}}$ are also homotopic paths in E.

$\Downarrow$

(3) $\forall$ simply connected topological space Y and a continuous map $f : Y \longrightarrow X$ sending point $y \in Y$ to $x \in X$ $\exists!$ continuous map $\tilde{f} : Y \longrightarrow E$ lifting f (i.e. $\pi \circ \tilde{f} = f$) and s.t. $\tilde{f}(y) = \tilde{x}$. More so, if X, Y are manifolds, f-regular $\Rightarrow \tilde{f}$ is regular (since E & X locally look the same).

An important role is played by the universal coverings $\pi : E \longrightarrow X$, i.e. those coverings of X for which E is simply connected. While they don't always exist for topological spaces, they do exist for manifolds X (which is exactly what we care about).

Classical Construction

Fix $x \in X$. Define $\tilde{X}$ as the set of all paths $\gamma : [0,1] \to X$ with $\gamma(0) = x$, up to homotopy. Note a natural map $\tilde{X} \xrightarrow{\pi} X$ given by $\pi(\gamma) := \gamma(1)$. Now $\forall y \in X$, since X is a manifold we can pick a small nbhd $y \in U \subseteq X$ which is simply connected. Then, $\pi^{-1}(U) \simeq U \times F$ where $F = \pi^{-1}(y) = \{$homotopy classes of paths from x to $y\}$. (with the map $U \times F \xrightarrow{\sim} \pi^{-1}(U)$ given by concatenation of paths). Gluing topologies on each $\pi^{-1}(U)$ we obtain a topology on $\tilde{X}$.

Excercise (*): a) Verify that $\tilde{X}$ is simply connected (hint: use homotopy lifting property) i.e. $\tilde{X}$ is a universal cover.

b) Verify the universal property of $\tilde{X}$, i.e. $\forall$ covering $\overset{\text{path-connected}}{E} \xrightarrow{p} X$ there is a covering $\tilde{X} \xrightarrow{\tilde{\pi}} E$ s.t. $p \circ \tilde{\pi} = \pi$.

c) Deduce that a universal covering is unique up to isomorphisms.

Notation: $\underbrace{\pi_1(X, x)}_{\text{fundamental group}} = \{$homotopy classes of closed paths from x to $x\}$
fundamental group (group operation is concatenation of paths)

Then: $\pi_1(X, x)$ acts on $\pi^{-1}(x)$ for any covering $\pi : E \to X$. Moreover, this action is free if E is a universal cover.