Lecture #3

- Discuss the notions of:
  - universal coverings (+ there classical construction)
  - fundamental group $\pi_1(X, x)$

  from the end of page 5 of Lecture #2.

- After this general detour to the theory of coverings, let us return to realm of Lie groups. Let G be a connected (real / complex) Lie group and $\widetilde{G}$ be the universal covering of G, so that $\widetilde{G} = \{\text{homotopy classes of paths } \gamma : [0,1] \to G \mid \gamma(0) = \underset{\text{unit of } G}{1}\}$

By above: $\widetilde{G}$ is a manifold. But it is also a group w.r.t. the following product:

$$\boxed{(\gamma_1 \cdot \gamma_2)(t) := \gamma_1(t) \underset{\uparrow \text{ product in } G}{\cdot} \gamma_2(t) \quad \text{for} \quad t \in [0,1]}$$

Lemma 1: $\widetilde{G}$ is a simply-connected Lie group, and the covering $\pi : \widetilde{G} \to G$ is a homomorphism of Lie groups.

▶ The only non-trivial part of this result is that $\widetilde{G}$ – Lie gp, i.e. $\widetilde{m} : \widetilde{G} \times \widetilde{G} \to \widetilde{G}$ – regular. But to this end, we note the commutative diagram

$$\begin{tikzcd} \widetilde{G} \times \widetilde{G} \arrow[r, "\widetilde{m}"] \arrow[d, "\pi \times \pi"'] & \widetilde{G} \arrow[d, "\pi"] \\ G \times G \arrow[r, "m"] & G \end{tikzcd}$$

Thus, $\widetilde{m}$ is a lifting of the map $m \circ (\pi \times \pi) : \underbrace{\widetilde{G} \times \widetilde{G}}_{\text{simply-connected}} \to G$. Hence, by part (3) of the Homotopy Lifting Property (from the end of Lecture 2), $\widetilde{m}$ is regular! ■

Basic Example: The universal cover of the Lie group $(S^1, \cdot)$ is $(\mathbb{R}, +)$.

Moreover, the universal covering $\widetilde{G}$ of G is just a central extension of G, due to

Lemma 2: $\underbrace{\mathrm{Ker}(\pi : \widetilde{G} \to G)}_{\simeq \pi_1(G, 1) \text{ by general property of universal coverings.}}$ is a central subgroup of $\widetilde{G}$

Exercise: a) Show that every discrete normal subgroup of a connected Lie gp is central.

b) Deduce the proof of Lemma 2 from part a).

Upshot: Last time we explained how the case of any Lie group can be reduced to connected ones, while by above we can further reduce to simply-connected ones.

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- To prove Prop 1 below, we start with the following definition:

Def 1: A closed Lie subgroup of a (real / complex) Lie group G is a subgroup H which is also an embedded submanifold.

recall: that means we have an immersion $X \xrightarrow{f} G$ with $\mathrm{image}(f) =$ our subgp H such that $f: X \to H$ is a homeomorphism.

!Note: In the above definition, we do not require H to be closed. However, the above terminology is justified by the following result:

Lemma 3: A closed Lie subgroup H of G is closed in G.

Exercise: a) Show that the closure $\overline{H}$ of H in G is a subgroup of G.
b) Show that each coset Hx for $x \in \overline{H}$ is open and dense in $\overline{H}$
c) Deduce that $\overline{H} = H$, i.e. H is closed in G.

The reverse is also true, but is much harder (so we'll just state it):

Theorem: Any closed subgroup of a real Lie group G is a closed Lie subgroup.

As an important application of Lemma 3, we have ("local-to-global"):

Proposition 1: a) If G is a connected Lie group and $\mathcal{U}$ is a nbhd of $1 \in G$, then $\mathcal{U}$ generates the entire G.
b) If $f: G \to H$ is a homomorphism of Lie groups, H is connected, and $d_1 f: T_1 G \twoheadrightarrow T_1 H$ is surjective, then f is surjective

▶ a) Let $K = \langle \mathcal{U} \rangle$ denote the subgp of G generated by $\mathcal{U}$. Then $\forall k \in K$, $k \cdot \mathcal{U} \subseteq K$, hence K is open in G. Therefore, K is an embedded submanifold of G, hence, K is a closed Lie subgp of G (see Def 1). But then K is also closed by Lemma 3.
As G is connected and $\emptyset \neq K$ - open & closed $\Rightarrow K = G \Rightarrow \mathcal{U}$ generates G

b) Surjectivity of $d_1 f: T_1 G \to T_1 H$ implies that $f(G)$ contains a nbhd of $1 \in H$ (here, we actually use the implicit function theorem). Thus, by part a), and the fact that $f(G)$ - subgroup, we get f - surjective. ∎

- One may ask how to obtain surjections of Lie gps from Prop 1(b).
To this end, recall that for discrete subgps $H \leq G$ one has the notion of coset space G/H, which is equipped with a gp structure if H is normal.
Let's now see how this can be adapted to the realm of Lie groups.

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Def 2: A regular map of manifolds $\pi : E \to B$ is called a *fiber bundle* with *base* B, *total space* E, and *fiber* being a manifold F if $\forall x \in B$ has a neighborhood $\mathcal{U}$ s.t. $\pi^{-1}(\mathcal{U}) \simeq \mathcal{U} \times F$ with π being projection $\mathcal{U} \times F \to \mathcal{U}$.

Note that for $\dim(F)=0$, this recovers the notion of covering (Lecture 2).

Here is the key result we care about in the context of Lie groups:

Proposition 2: a) Let G be a (real/complex) Lie group of dimension n and $H \subseteq G$ a closed Lie subgroup of dimension k. Then the coset space G/H (called a *homogeneous space* of G) has a natural structure of a manifold of dimension $n-k$, and the map $\pi : G \to G/H$ is a fiber bundle with fiber H.

b) The tangent space at $\bar{1} = \pi(1)$ is given by $T_{\bar{1}}(G/H) = T_1 G / T_1 H$.

c) If moreover H is normal in G, then G/H is a (real/complex) Lie gp

► Pick $\bar{g} \in G/H$ and choose any $g \in \pi^{-1}(\bar{g})$. Then $g \cdot H = L_g(H)$ is an embedded submnfd of G (i.e. image of H under L_g). Choose a small (so that we can work in local chart) transversal submanifold $M \subseteq G$ passing through g, i.e. $T_g G = T_g(gH) \oplus T_g M$.

In particular, $\dim M = n-k$. Choosing M small enough ensures that

$$\boxed{M \cdot H = \{m \cdot h \mid m \in M, h \in H\} - \text{open in } G}$$

To this end, we again apply the inverse function theorem to $m: M \times H \to G$

Set $\bar{M} := \pi(M)$. Then $\bar{M}$ is an open nbhd of $\bar{g} \in G/H$ (as $\pi^{-1}(\bar{M}) = M \cdot H$ is open). It's also clear that $\pi : M \to \bar{M}$ is a homeomorphism. This gives a local chart near $\bar{g} \in G/H$. The transition maps are regular and so we get a structure of a manifold on G/H (independent of the choices made).

Furthermore, since $M \times H \to M \cdot H$ is a diffeomorphism, π is a fiber bundle with the fiber H. This proves part a).

Evoking $T_g G = T_g(gH) \oplus T_g M$, we also get $T_{\bar{1}}(G/H) \simeq T_1 G / T_1 H$. This proves part b).

Finally, we leave part c) as a simple exercise (it suffices to check regularity of multiplication at 1) ■

Exercise (*): a) Prove part c) of Prop 2

b) Verify that manifold structure constructed above is independent of both $g \in \pi^{-1}(\bar{g})$ and the small transversal submanifold M.

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As an immediate corollary of Prop 2, we get:

Corollary 1: Let H be a closed Lie subgroup of a Lie group G.

a) If H is connected, then the map $\pi_0(G) \to \pi_0(G/H)$ between the sets of connected components is a bijection. In particular, if $H, G/H$ are connected then so is G.

b) If both H, G are connected, then there is an exact sequence of fundam. gps:
$$\pi_1(H) \to \pi_1(G) \to \pi_1(G/H) \to \{1\}$$

Recall: exact here means that $\mathrm{Ker}(\pi_1(G) \to \pi_1(G/H)) = \mathrm{Im}(\pi_1(H) \to \pi_1(G))$
and $\pi_1(G) \to \pi_1(G/H)$ is surjective

The above is just the very end of the famous long exact sequence of (higher) homotopy groups of a fibration. It's a good exercise to prove the above:

Exercise(*): Prove Corollary 1.

[A nice application of this result is the computation of π_1 for classical gps. – to be discussed later on]

- For many purposes, the notion of a closed Lie subgroup (Def 1) is quite restrictive.

[Example ("Irrational winding on the torus"): Consider $\mathbb{R} \to \underset{\text{torus}}{T^2} = \mathbb{R}^2/\mathbb{Z}^2$
Then the image is dense in T^2. $\qquad t \mapsto (t \bmod \mathbb{Z}, \pi \cdot t \bmod \mathbb{Z})$]

For this reason, it is useful to consider the following notion:

Def 3: A Lie subgroup of a Lie group G is a subgroup H which is also an immersed submanifold (i.e. there is a mnfd structure on H and immersion $H \hookrightarrow G$)
↑ does not need to be an embedded submanifold or a closed subset

In particular, the above "irrational winding of the torus" or "$(\mathbb{Q},+) \subseteq (\mathbb{R},+)$" are Lie subgps

The relation to the above discussion is provided by the following result:

Proposition 3: Let $f: G_1 \to G_2$ be a homomorphism of Lie groups. Then, $H := \mathrm{Ker}(f)$ is a normal closed Lie subgroup in G_1, while $\mathrm{Im}(f)$ is a Lie subgroup in G_2 (Def 3). If furthermore $\mathrm{Im}(f)$ is an embedded submanifold of G_2, then it is a closed Lie subgroup in G_2 and f gives rise to an isomorphism of Lie groups $G_1/H \simeq \mathrm{Im}(f)$

We shall prove this result after some basics on Lie algebras.

• We conclude today's lecture by discussing the primary reason why Lie gps are so important (arising as symmetries of various geometric objects).

Def 4: Let X be a manifold, G be a Lie group, and $a: G\times X \to X$ be a regular map. We call this a *regular (left) action of G on X* if $(g_1\cdot g_2).x = g_1.(g_2.x)$, i.e.

$$a(m(g_1,g_2),x) = a(g_1, a(g_2,x)) : G\times G\times X \to X$$

In other words $\forall g\in G$ we have a diffeomorphism $\rho(g) = a(g,\cdot)\in \mathrm{Diff}(X)$ with $\boxed{\rho(1)=\mathrm{Id} \quad \text{and} \quad \rho(g_1g_2) = \rho(g_1)\circ\rho(g_2)}$

Examples: a) $GL_n(\mathbb{R}) \curvearrowright \mathbb{R}^n$

b) $O_n(\mathbb{R}) \curvearrowright S^{n-1} = \text{unit sphere in } \mathbb{R}^n$

Closely related with the above notion is the notion of a representation:

Def 5: A *representation* of a (real or complex) Lie group G is a vector space V (complex if G is complex; real or complex if G is real) together with a group morphism $\rho: G\to \mathrm{End}(V)$. When V is finite-dimensional, we require ρ to be analytic, so that it is a morphism of Lie groups

A *homomorphism* of G-representations V and W is a linear map $f: V\to W$ that commutes with G-action, i.e. $A(\rho_V(g)v) = \rho_W(g)(Av)$ $\forall g\in G, v\in V$.

◦ Given $G\curvearrowright V$ one has a natural repr-n $G\curvearrowright V^*$, called the *dual repr-n*:

$$\boxed{\rho_{V^*}(g) = \rho_V(g^{-1})^* : V^*\to V^*}$$

◦ Given $G \overset{\rho_V}{\curvearrowright} V$ and $G\overset{\rho_W}{\curvearrowright} W$, one has the *tensor product repr-n* $G\curvearrowright V\otimes W$:

$$\boxed{\rho_{V\otimes W}(g) = \rho_V(g)\otimes\rho_W(g) : V\otimes W \to V\otimes W.}$$

◦ Given $G\overset{\rho_V}{\curvearrowright} V$ and subspace $W\subseteq V$ s.t. $\rho_V(g)W\subseteq W$ $\forall g$, one has a natural *quotient representation* $G\curvearrowright V/W$.

Finally, the most relevant to us is the *adjoint action* of any Lie group G on $\mathfrak{g} = T_1G$. To this end, consider the composition of the *left & right translations*

$$\begin{array}{ccc} G\times G \longrightarrow G & \quad\text{and}\quad & G\times G \longrightarrow G \\ (g,h)\longmapsto gh & & (g,h)\longmapsto hg^{-1} \end{array}$$

to get the *adjoint action* $G\times G \xrightarrow{Ad} G$ given by $Ad_g(x) = gxg^{-1}$. Since $Ad_g(1)=1$ $\forall g$, we get $d_1(Ad_g): \mathfrak{g}\to\mathfrak{g}$ that is commonly denoted just by Ad_g.