Lecture #4

- Last time, we discussed:
  - universal covering of connected Lie gps
  - {closed Lie subgps of a Lie gp G} $\longleftrightarrow$ {closed subgroups of G}
  - any connected Lie gp is generated by any nbhd of 1.
  - manifold structure on G/H (H-closed Lie subgp of G); fiber bundle $G \to G/H$
  - Lie subgps of a Lie gp G; and Ker / Im for any Lie gp homom.
  - actions and representations of Lie groups.

- Let me start by making a couple of important observations relevant to the end of previous lecture.
  - If $G \curvearrowright M$ is an action on a mfd, then it gives rise to a representation

$$\begin{cases} G \curvearrowright C^\infty(M) \text{ for real setup} \\ G \curvearrowright \mathcal{O}(M) = \{\text{holomorphic f-s}\} \text{ in complex setup} \end{cases} \quad \text{via} \quad \boxed{(g.f)(x) \overset{\text{def}}{=} f(g^{-1}.x)}$$

note the inverse

  - If $G \curvearrowright M$ as above, then we also get natural $G \curvearrowright \mathrm{Vect}(M) = \{\text{vector fields on } M\}$ and more general tensor fields on M

  (we shall discuss this next week)

  - Furthermore, if $x \in M$ is G-fixed (i.e. $g.x = x \ \forall g \in G$), then we get a representation $G \overset{\rho}{\curvearrowright} T_xM$ through $\rho(g) = g_* : T_xM \longrightarrow T_xM$ (and similarly various tensors)

  fin. dimensional, unlike $C^\infty(M)$, Vect_M

  - In particular, for any Lie group G there are 3 important actions $G \curvearrowright G$:
    - *Left action*: $G \times G \to G$, $(g,h) \mapsto gh$, so that $L_g(h) = g \cdot h$
    - *Right action*: $G \times G \to G$, $(g,h) \mapsto hg^{-1}$, so that $R_g(h) = h \cdot g^{-1}$
    - *Adjoint action*: $G \times G \to G$, $(g,h) \mapsto ghg^{-1}$, so that $\mathrm{Ad}(g)(h) = ghg^{-1} = L_g R_g(h)$

While left and right actions have no fixed points, we note that $1 \in G$ is preserved under the adjoint action, so by above we get a representation

$$G \curvearrowright T_1 G := \mathfrak{g} \quad \text{called an adjoint representation}$$

i.e. $\forall g \in G$, have $\boxed{\mathrm{Ad}(g) = d_1 \mathrm{Ad}(g) : \mathfrak{g} \to \mathfrak{g}}$

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- Given an action $G \curvearrowright X$ of a Lie gp G on a mnfd X, for every point $x \in X$, we have two classical notions associated:
  - orbit $Gx \subseteq X$ defined as $\{g.x \mid g \in G\} \subseteq X$
  - stabilizer $G_x \subseteq G$ defined as $\{g \in G \mid g.x = x\}$.

The following result is a Lie gp version of orbit-stabilizer thm for finite gps:

Proposition 1: The stabilizer G_x is a closed Lie subgroup in G, and the natural map $G/G_x \to X$, $\bar{g} \mapsto g.m$ (g - lift of $\bar{g}$) is an injective immersion with image Gx.

We shall sketch the proof of this result later on after discussing Lie algebras (note that "G_x is a closed Lie subgp in G" is a direct consequence of the harder more general result from Theorem in Lecture 3)

Corollary 1: The orbit $Gx \subseteq X$ is an immersed submanifold with tangent space $T_x(Gx) \simeq T_1(G)/T_1(G_x)$. If Gx is an embedded submanifold, then the map $G/G_x \to Gx$ is a diffeomorphism.

(note that Gx does not need to be closed, e.g. $\mathbb{C}^\times \curvearrowright \mathbb{C}$ by multiplication)

A special case of actions is when there is just 1 orbit, i.e. $G \curvearrowright X$ <u>transitively</u>. The common terminology is:

|| <u>Notation</u>: A G-<u>homogeneous space</u> is a manifold with a transitive action.

Thus, combining Corollary 1 with Proposition 2 from Lecture 3, we get:

Corollary 2: Let X be a G-homogeneous space and $x \in X$, then the map $G \to X$, $g \mapsto g.x$, is fiber bundle over X with fiber G_x.

Let us emphasize right away that the above result can be used to get a smooth structure on a set, endowed with a transitive action of a Lie group G; we discuss a few classical examples next.

Warning: In general, if $G \curvearrowright X$, there is no obvious way to define the quotient X/G so that it has nice properties. To this end, one either needs to impose some additional conditions on the action, or instead use an alternative approach of GIT quotients from alg. geometry.
<u>Example</u>: The naive quotient $\underbrace{\mathrm{Mat}_{n\times n}(\mathbb{C})/GL_n(\mathbb{C})}_{\text{adjoint action}}$ is <u>not</u> even Hausdorff!

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- Examples:
  - $SO(n,\mathbb{R}) \curvearrowright \underset{\uparrow \text{unit sphere}}{S^{n-1}} \subseteq \mathbb{R}^n$ transitively with stabilizers $\simeq SO(n-1,\mathbb{R})$.

    Hence, we get a fiber bundle
$$\begin{array}{ccc} SO(n-1,\mathbb{R}) & \longrightarrow & SO(n,\mathbb{R}) \\ & & \downarrow \\ & & S^{n-1} \end{array}$$

    In particular: $SO(2,\mathbb{R}) \simeq S^1$
    $SO(3,\mathbb{R})$ fibers over S^2 with fiber S^1

  - $SU(n) \curvearrowright S^{2n-1} \subseteq \mathbb{C}^n$ transitively with stabilizers $\simeq SU(n-1)$.

    Hence, we get a fiber bundle
$$\begin{array}{ccc} SU(n-1) & \longrightarrow & SU(n) \\ & & \downarrow \\ & & S^{2n-1} \end{array}$$

    <u>In particular</u>: $SU(2) \simeq S^3$

    [<u>Remark/Exercise</u>: We also have $SU(2) \curvearrowright \mathbb{C}P^1 \simeq S^2$ transitively with stabilizers $\simeq U(1) \simeq S^1$
    This recovers <u>Hopf fibration</u>
$$\begin{array}{ccc} S^1 & \longrightarrow & S^3 \\ & & \downarrow \\ & & S^2 \end{array}$$
    ]

  - Recall that (complete/full) flag in $\mathbb{R}^n$ is a sequence of subspaces
$$\{0\} \subset V_1 \subset V_2 \subset \dots \subset V_{n-1} \subseteq V_n = \mathbb{R}^n \quad \text{with } \dim_{\mathbb{R}}(V_i) = i$$
    Let $\mathcal{F}_n(\mathbb{R})$ denote the set of all flags. Then, we have a natural action $GL_n(\mathbb{R}) \curvearrowright \mathcal{F}_n(\mathbb{R})$ which is transitive! Moreover, the stabilizers are $\simeq B_n(\mathbb{R})$ - Borel subgp of upper-triangular matrices in $GL_n(\mathbb{R})$. Thus:
$$\begin{array}{ccc} B_n(\mathbb{R}) & \longrightarrow & GL_n(\mathbb{R}) \\ & & \downarrow \\ & & \mathcal{F}_n(\mathbb{R}) \end{array}$$
    Therefore, $\mathcal{F}_n(\mathbb{R})$ has a natural mnfd structure and is called <u>flag variety</u>.

    [<u>Exercise</u>: Generalize to partial flags (for any sequence $0 < d_1 < d_2 < \dots < d_k = n$)
$$\{0\} \subseteq V_1 \subset V_2 \subseteq \dots \subseteq V_{k-1} \subseteq V_k = \mathbb{R}^n \quad \text{with } \dim_{\mathbb{R}}(V_i) = d_i$$
    ]

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• Classical Lie groups

We shall first start discussion of (Lie gps) $\leftrightarrow$ (Lie algebras) for the subgps of the general linear groups $GL_n(\mathbb{R})$ or $GL_n(\mathbb{C})$ that naturally arise in linear algebra

- $GL_n(\mathbb{K})$ with $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$.
- $SL_n(\mathbb{K})=\{A\in GL_n(\mathbb{K})\mid \det A=1\}$ – "special linear group"
- $O(n,\mathbb{K})=\{A\in GL_n(\mathbb{K})\mid A\cdot A^T=1\}$ – "orthogonal group"
- $SO(n,\mathbb{K})=\{A\in O(n,\mathbb{K})\mid \det A=1\}$ – "special orthogonal group".

  Note: $O(n,\mathbb{K})$ is the gp of matrices that preserve a nondegenerate quadratic form $Q(x_1,\ldots,x_n)=x_1^2+\ldots+x_n^2$ or the correspondy bilinear form $B(x_1,\ldots,x_n;y_1,\ldots,y_n)=x_1y_1+\ldots+x_ny_n$.

- "pseudo-orthogonal" $O(p,q)=\{A\in GL_{n=p+q}(\mathbb{R})\mid$ preserve a nondegenerate quadratic form of signature (p,q) $Q(x_1,\ldots,x_{p+q})=x_1^2+\ldots+x_p^2-x_{p+1}^2-\ldots-x_{p+q}^2\}$
- $SO(p,q)=\{A\in O(p,q)\mid \det A=1\}$ ← "special pseudo-orthogonal group"
- $Sp(2n,\mathbb{K})=\{A\in GL_{2n}(\mathbb{K})\mid$ preserve nondegenerate skew-symmetric bilinear form, all of which are equivalent to $B(x_1,\ldots,x_{2n};y_1,\ldots,y_{2n})=\sum_{i=1}^n x_iy_{i+n}-\sum_{i=1}^n x_{i+n}y_i\}$
- "unitary group" $U(n)=\{A\in GL_n(\mathbb{C})\mid A\cdot \underbrace{A^*}_{\bar{A}^T}=1\}$ and more generally

  $U(p,q)=\{A\in GL_{n=p+q}(\mathbb{C})\mid$ preserve a nondegenerate Hermitian quadratic form of signature (p,q), i.e. $Q(x_1,\ldots,x_{p+q})=|x_1|^2+\ldots+|x_p|^2-|x_{p+1}|^2-\ldots-|x_{p+q}|^2\}$

  ↑ "pseudo-unitary group"

- $SU(n)$ and $SU(p,q)$ ← add $\det A=1$

  ↑ "special unitary" ↑ "special pseudo-unitary"

- ⋮ some other groups, which we will postpone till later.

In fact, all these are Lie groups! This is not surprising in view of the Theorem of Lecture 3 which does not really provide much insight here (in particular, even their dimensions will not be obtained). Instead, we would really like to get local coordinates near 1 (note that applying the implicit function theorem is infeasible either).

To this end, we recall the <u>exponential</u> and <u>logarithmic</u> maps from linear algebra

The exponential map is an analytic map $\exp : \mathfrak{gl}_n(\mathbb{K}) \to \mathfrak{gl}_n(\mathbb{K})$ given by

$$\boxed{\exp(x) = \sum_{n\geq 0} \frac{x^n}{n!}} \leftarrow \text{for any } x \in \mathfrak{gl}_n(\mathbb{K})$$

(recall how to prove that it's regular and how to practically compute $\exp(x)$).

Likewise, one can also use Taylor series for $\log(1+x)$ to define the logarithmic map, defined only in a neighborhood of $1_n \in \mathfrak{gl}_n(\mathbb{K})$:

$$\boxed{\log(A) = \sum_{n\geq 1} (-1)^{n-1} \frac{(A-1)^n}{n} = -\sum_{n\geq 1} \frac{(1-A)^n}{n}} \leftarrow \text{for } A \in \begin{array}{l}\text{nbhd of}\\ 1_n \in \mathfrak{gl}_n(\mathbb{K})\end{array}$$

The properties of these maps are summarized in the following exercise:

Exercise: Prove the following properties:

1) exp & log are mutually inverse when defined, i.e. $\exp(\log A) = A$, $\log(\exp(x)) = x$
2) exp & log are conjugation invariant, i.e. $\exp(BxB^{-1}) = B\exp(x)B^{-1}$
   $\log(BAB^{-1}) = B\log(A)B^{-1}$
3) $\det(\exp(x)) = \exp(\mathrm{Tr}(x))$ and $\log(\det A) = \mathrm{Tr}(\log A)$ in small nbhd of 1_n
4) $d_0 \exp = \mathrm{Id}$ and $d_1 \log = \mathrm{Id}$
5) if $xy = yx$, then $\exp(x+y) = \exp(x)\exp(y)$
   if $AB = BA$, then $\log(AB) = \log(A) + \log(B)$ in small nbhd of 1_n
6) $\forall x \in \mathfrak{gl}_n(\mathbb{K})$, the map $\gamma_x : \mathbb{K} \to GL_n(\mathbb{K})$ is a homom. of Lie groups.
   $t \mapsto \exp(tx)$

With this in mind, we can prove:

Theorem 1: For each classical group G from the list on p.4, there is a vector subspace $\mathfrak{g} \subseteq \mathfrak{gl}_n(\mathbb{K})$, s.t. for some open nbhds $1 \in \mathcal{U} \subseteq GL_n(\mathbb{K})$ and $0 \in \mathfrak{u} \subseteq \mathfrak{gl}_n(\mathbb{K})$, the exp & log maps

$$\mathcal{U} \cap G \underset{\exp}{\overset{\log}{\rightleftarrows}} \mathfrak{u} \cap \mathfrak{g}$$

are mutually inverse.

$\Downarrow$

Corollary 3: Each classical group is a Lie group with $T_1 G = \mathfrak{g}$ (from Thm 1), so that $\dim G = \dim \mathfrak{g}$. Moreover, every basis of $\mathfrak{g}$ gives rise to local coordinates near $1 \in G$

Terminology: $\mathfrak{g} = T_1 G$ is called the Lie algebra of the Lie group G.

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Before presenting the proof of Thm 1, let's give the key definition:

Def 1: A Lie algebra over a field $\mathbb{K}$ is a vector space $\mathfrak{g}$ over $\mathbb{K}$ equipped with a bilinear operation $\underbrace{[\,,\,]}_{\text{Lie bracket}}: \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$ s.t.:

1) $[X,X]=0$

2) Jacobi identity $\boxed{[[X,Y],Z]+[[Y,Z],X]+[[Z,X],Y]=0 \quad \forall X,Y,Z\in\mathfrak{g}}$

Note: If char $\mathbb{K}\neq 2$, then 1) $\Leftrightarrow$ $[X,Y]+[Y,X]=0$ i.e. $[\,,\,]$ - skew-symmetric

General result that we'll prove next time is that:

$$\boxed{\forall \text{ Lie group } G \text{ over } \underset{\mathbb{R} \text{ or } \mathbb{C}}{\mathbb{K}} \rightsquigarrow T_1G \text{ has a natural structure of a Lie algebra over } \mathbb{K}.}$$

Moreover, for classical groups from p.4, the Lie bracket is the restriction of the usual commutator

$$\boxed{\begin{aligned}[\,,\,]: \mathfrak{gl}_n(\mathbb{K})\times\mathfrak{gl}_n(\mathbb{K}) &\longrightarrow \mathfrak{gl}_n(\mathbb{K})\\ (x,y) &\longmapsto [x,y]=x\cdot y-y\cdot x\end{aligned}}$$

The fact that $\mathfrak{g}$ is closed w.r.t. $[\,,\,]$ will follow from:

$$\boxed{\exp(x)\cdot\exp(y) = \exp\left(x+y+\frac{[x,y]}{2}+\text{h.o.t}\right)}$$

check it! (easy exercise)

After the above short detour, let's present the proof of the Theorem

(Proof of Theorem 1)

- $G=GL_n(\mathbb{K}) \rightsquigarrow$ pick $\boxed{\mathfrak{g}=\mathfrak{gl}_n(\mathbb{K})}$. Remains to apply the Exercise on properties of exp & log.
- $G=SL_n(\mathbb{K}) \rightsquigarrow$ Assume A is in nbhd of 1_n so that we can write $A=\exp(x)$ with $x=\log(A)$.
  Then: $1=\det A=\det(\exp(x))=\exp(\mathrm{tr}(x)) \Rightarrow \mathrm{tr}(x)=0$ (in complex setting, assume x is in small nbhd of 0)
  Hence $\boxed{\mathfrak{g}=\mathfrak{sl}_n(\mathbb{K})=\{x\in\mathfrak{gl}_n(\mathbb{K})\mid \mathrm{tr}(x)=0\}}$ works.
- $G=O(n,\mathbb{K}) \rightsquigarrow$ Assume A is in nbhd of 1_n, so that $A=\exp(x)$ with $x=\log(A)$
  $A\in O(n,\mathbb{K}) \Leftrightarrow A^t=A^{-1} \Leftrightarrow \exp(x)^t=\exp(x)^{-1} \Leftrightarrow x^t=-x$
  Hence: $\boxed{\mathfrak{g}=\mathfrak{o}_n(\mathbb{K})=\{x\in\mathfrak{gl}_n(\mathbb{K})\mid x+x^t=0\}}$
- $G=SO(n,\mathbb{K}) \rightsquigarrow$ In this case, on top of $x+x^t=0$ we also add $\mathrm{tr}(x)=0$
  However, the latter readily follows from the former as $\mathrm{tr}(x+x^t)=2\,\mathrm{tr}(x)$
  So: $\boxed{\mathfrak{g}=\mathfrak{o}_n(\mathbb{K})=\mathfrak{so}_n(\mathbb{K})}$.
  same Lie algebras

  Note: The gp $SO(n,\mathbb{K})$ is just a connected component of $O(n,\mathbb{K})$

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(Continuation of the proof)

- $G=U(n) \rightsquigarrow$ similarly to $O(n)$, we get $\boxed{\mathfrak{g}=u_n=\{x\in gl_n(\mathbb{C})\mid x+\bar{x}^T=0\}}$ ← anti-hermitian matrices

  $G=SU(n) \rightsquigarrow \boxed{\mathfrak{g}=su_n=\{x\in gl_n(\mathbb{C})\mid x+\bar{x}^T=0,\ tr(x)=0\}} \neq u_n$

- For the remaining gps from p.4, this is a homework exercise ∎

Exercise: a) Compute $\mathfrak{g}$ for the remaining groups
$O(p,q)$, $SO(p,q)$, $Sp(2n,\mathbb{K})$, $U(p,q)$, $SU(p,q)$

Hint: it is convenient to present the answer in terms of the matrix associated to the corresponding bilinear form.

b) Compute $\text{dimensions}_{\mathbb{R}}$ of all classical groups