Lecture #5

- Last time, we discussed
  - stabilizer-orbit thm for Lie gps
  - homogeneous spaces and their mnfd structure
  - classical gps and Lie gp structure on them via exp-map
  - notion of Lie algebras in general

Let me start by making a few remarks

1) The Lie alg-s of corresponding to $O(n,\mathbb{K})$ and $SO(n,\mathbb{K})$ coincide:
$$\begin{aligned} \mathfrak{o}_n(\mathbb{K}) &= \{x \in \mathfrak{gl}_n(\mathbb{K}) \mid x + x^T = 0\} \\ &\parallel \\ \mathfrak{so}_n(\mathbb{K}) &= \{x \in \mathfrak{gl}_n(\mathbb{K}) \mid x + x^T = 0,\ \mathrm{tr}(x) = 0\} \end{aligned}$$
This is not surprising as $SO(n,\mathbb{K})$ is a connected component of $O(n,\mathbb{K})$

2) In contrast, $\mathfrak{su}_n \neq \mathfrak{u}_n$.

3) To describe Lie alg-s $\mathfrak{g}$ for classical gps corresponding to bilinear form $B(-,-)$, you may wish to take $\frac{d}{dt}\Big|_{t=0} B(e^{tx}u, e^{tx}v)$
(you'll see it in Hwk 2, Problem 8a)

4) The fiber bundles $\begin{array}{c} H \to G \\ \downarrow \\ X = G/H \end{array}$ may be used to compute $\pi_1(X)$ via Hwk 2, Problem 5(b)
(you'll get it in Hwk 3)

- Goal for today: Analogue of exp-map for general Lie groups.

To do so, we shall first recall the notion of vector fields, which are just sections of tangent bundles.

Def 1: Let X be a real mnfd, and let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A $\mathbb{K}$-vector bundle of rank n on X is a mnfd E with $\pi: E \twoheadrightarrow X$ and $\mathbb{K}$-vector space structure on each $\pi^{-1}(x)$ so that $\forall x \in X$ $\exists$ open nbhd $x \in U \subseteq X$ with $\boxed{\pi^{-1}(U) \underset{g_U}{\xrightarrow{\sim}} U \times \mathbb{K}^n}$ where projection is on 1st component, fiber's vector space structure is on the 2nd cpt (compare to fiber bundes in general)

Covering X by such open charts $\{U_\alpha\}$ we get transition maps which give rise to:
$$\boxed{\text{clutching functions } U_\alpha \cap U_\beta \xrightarrow{h_{\alpha\beta}} GL_n(\mathbb{K}) \text{ s.t. } g_{U_\alpha} \circ g_{U_\beta}^{-1}: (x, v) \mapsto (x, h_{\alpha\beta}(v))}$$
for any pair $U_\alpha \cap U_\beta \neq \emptyset$. Here, $g_{U_\alpha}: \pi^{-1}(U_\alpha) \xrightarrow{\sim} U_\alpha \times \mathbb{K}^n$ as in Def 1. Moreover:
$$\boxed{h_{\alpha\beta} = h_{\beta\alpha}^{-1} \text{ and } h_{\alpha\beta} \circ h_{\beta\gamma} = h_{\alpha\gamma}}$$

Exercise: Revert the above construction, i.e. to any open cover $\{U_\alpha\}_{\alpha\in I}$ of X with a collection $\{h_{\alpha\beta}\}_{\alpha,\beta\in I}$ satisfying above properties construct the corresponding vector bundle.

Example (important for what follows): Take an atlas $\{(U_\alpha,\varphi_\alpha)\}$ defining mnfd structure on X, and define $h_{\alpha\beta}(x)$ as the differential of $\tau_{\alpha\beta}$ at x, i.e.

$$\boxed{h_{\alpha\beta}(x) = d_{\varphi_\beta(x)}(\tau_{\alpha\beta}) \quad \text{with} \quad \tau_{\alpha\beta} = \varphi_\alpha\circ\varphi_\beta^{-1} : \varphi_\beta(U_\alpha\cap U_\beta)\to\varphi_\alpha(U_\alpha\cap U_\beta)}$$

Exercise: Check the above compatibility conditions for these $h_{\alpha\beta}$.

This defines a vector bundle, called the tangent bundle $TX\xrightarrow{\pi}X$, so that $\pi^{-1}(x) = T_xX$, the tangent space of X at point x.

Def 2: A section of a vector bundle $\pi: E\twoheadrightarrow X$ is a map $s: X\hookrightarrow E$ s.t. $\pi\circ s = Id_X$

Notation: $\Gamma(X,E) = \{$vector space of all sections of $E\twoheadrightarrow X\}$

More generally, $\Gamma(U,E) = \{$sections of $E\twoheadrightarrow X$ over $U\}$.

Exercise: Show that a rank n $\mathbb{K}$-vector bundle $E\xrightarrow{\pi}X$ is trivial (i.e. $E\simeq X\times\mathbb{K}^n$ over X) iff $\exists\, s_1,\dots,s_n\in\Gamma(X,E)$ s.t. $\{s_1(x),\dots,s_n(x)\}$ – basis of $\pi^{-1}(x)$ $\forall x$.

Finally, we can now define the vector fields:

Def 3: A vector field on X is a section of the tangent bundle $TX\to X$

Notation: $Vect(X)$.

Down-to-earth, we can think of a vector field V locally as expression $V=\sum_{i=1}^n v_i(x)\partial/\partial x_i$ in local coordinates $\{x_i\}$, where v_i is a regular f-n, and switching to a different chart with local coordinates $\{x_i'\}$, we get

$$\boxed{v_i' = \sum_{j=1}^n \frac{\partial x_i'}{\partial x_j} v_j}$$

Note: Every such V gives rise to a derivation of $\mathcal{O}(U)$ and germ $\mathcal{O}_x$.
Vice versa, derivations of $\mathcal{O}(U)$ compatible with $U'\subseteq U$ give rise to v. field.

Finally, if a Lie group G acts on mnfd X, then it also acts on TX

$$\boxed{G\curvearrowright X \rightsquigarrow G\curvearrowright TX \rightsquigarrow G\curvearrowright Vect(X)}$$

In particular, $L_g, R_g: G\curvearrowright G$ give rise to $G\curvearrowright Vect(G)$, also denoted L_g or R_g.

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Def 4: a) A vector field $V \in Vect(G)$ is left-invariant if $L_g V = V \ \forall g \in G$.

b) — // — is right-invariant if $R_g V = V \ \forall g \in G$.

c) — // — is bi-invariant if $L_g V = V = R_g V \ \forall g \in G$

Lemma 1: a) The map $V \mapsto V(\underset{\text{unit of } G}{1})$ gives rise to a vector space isomorphism

$$\{\text{left-invariant vector fields on } G\} \xrightarrow{\sim} T_1 G.$$

b) Similarly for right-inv. vector fields

▷ a) $\forall x \in \mathfrak{g} = T_1 G$, define $V \in Vect(G)$ via $V(g) = L_g(x)$, thus spreading x all over G. It's clear that this V is left-invariant. Moreover, it's unique such with $V(1) = x$.

b) Analogous, but use R_g instead of L_g ∎

Exercise: Show that $V \mapsto V(1)$ gives rise to a vector space isom.

$$\{\text{bi-invariant vector fields on } G\} \xrightarrow{\sim} (T_1 G)^{Ad\,G} = \{\underset{T_1 G}{x} \mid Ad_g(x) \overset{\forall g}{=} x\}$$

Exercise: Verify that if $G = GL_n(\mathbb{R}) \subseteq Mat_{n\times n}(\mathbb{R})$ so that each $T_g G$ is canonically identified with $Mat_{n\times n}(\mathbb{R})$, then the left-invariant vector field on G is given by $g \mapsto gv$ (usual product of matrices); state analogous f-la for right-invariant.

Corollary 1: For any Lie group G, the tangent bundle $TG \to G$ is trivial.

Remark: The above can be generalized from vector fields to tensor fields. To this end, let T^*X be the dual of the TX, called cotangent bundle. A tensor field of rank (n,m) on mnfd X is a section of $(TX)^{\otimes n} \otimes (T^*X)^{\otimes m}$. In particular, tensor fields of rank $(0,1)$ are called differential 1-forms (Locally they can be written as $\Omega = \sum_{i=1}^n \omega_i dx_i$, but changing coordinates we have $\omega_i' = \sum_j \frac{\partial x_j}{\partial x_i'} \omega_j$, cf. v_i' vs v_j. In particular, $\forall f \in \mathcal{O}(X)$ get 1-form $df = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i$)

Then, similarly to Lemma 1, we have

$$\left\{\begin{matrix}\text{left (or right) invariant} \\ \text{tensor fields of rank } (n,m) \text{ on } G\end{matrix}\right\} \simeq (T_1 G)^{\otimes n} \otimes (T_1 G)^{*\otimes m}$$

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The key in the general construction of the exponential map $T_1G =: \mathfrak{g} \to G$ is the construction of one-parameter subgroups:

> **Proposition 1**: Let G be a real Lie group and set $\mathfrak{g} := T_1G$.
> Then $\forall x \in \mathfrak{g}$, $\exists!$ morphism of Lie groups $\gamma = \gamma_x : \mathbb{R} \to G$ with $\underbrace{\gamma'(0)}_{= d_1\gamma(1)} = x$

- For classical groups, such γ was constructed last time via $\gamma(t) = \exp(tx)$.
In general, this doesn't make sense as there is no product on T_1G. However, similarly to calculus we note that such γ should satisfy
$$\boxed{\gamma'(t) = \gamma(t)\cdot\gamma'(0) = (L_{\gamma(t)})_*\gamma'(0)} \quad (*)$$
(to see this, write $\gamma(t+s) = \gamma(t)\cdot\gamma(s)$ and take $\frac{d}{ds}\big|_{s=0}$ of both sides)

By existence & uniqueness theorem for ODE with initial condition, the eq-n $(*)$ has a unique solution with $\gamma'(0) = x$ in the nbhd of 0, i.e. for $-\varepsilon < t < \varepsilon$.

- First, we claim that this γ satisfies group morphism on this nbhd, i.e.
$$\boxed{\gamma(t+s) = \gamma(t)\cdot\gamma(s) \text{ when } s,t,s+t \in (-\varepsilon,\varepsilon)} \quad (**)$$
Indeed, fix t and consider $\begin{cases}\gamma_1(s) := \gamma(t+s) \\ \gamma_2(s) := \gamma(t)\cdot\gamma(s)\end{cases}$. Then $\left.\begin{matrix}\gamma_1(0) = \gamma_2(0) = \gamma(t) \\ \gamma_1'(0) = \gamma'(t) \overset{(*)}{=} \gamma_2'(0)\end{matrix}\right\} \Rightarrow$

$\Rightarrow$ by uniqueness $\gamma_1 \equiv \gamma_2 \Rightarrow \gamma(t+s) = \gamma(t)\gamma(s)$ as above in $(**)$

- Also note that $\boxed{\gamma(t).x = x.\gamma(t) \text{ for } t \in (-\varepsilon,\varepsilon)} \quad (***)$

To this end, compute $\frac{d}{ds}\big|_{s=0}$ in $\gamma(s)\gamma(t) = \gamma(t+s) = \gamma(t)\gamma(s)$

- It remains to extend γ from $(-\varepsilon,\varepsilon)$ to $\mathbb{R}$, i.e. $\gamma : \mathbb{R} \to G$.
(There is at most one way to do so to preserve gp homomorphism, and we need to check consistency $\forall t,s$.)

We do so by extending iteratively from $(-2^k\cdot\varepsilon, 2^k\cdot\varepsilon)$ to $(-2^{k+1}\varepsilon, 2^{k+1}\varepsilon)$ via:
$$\boxed{\gamma(t) = \gamma(\tfrac{t}{2})\cdot\gamma(\tfrac{t}{2}) \quad \forall\ t \in (-2^{k+1}\varepsilon, 2^{k+1}\varepsilon)} \leftarrow \gamma(\tfrac{t}{2}) \text{ is defined as } \tfrac{t}{2} \in (-2^k\varepsilon, 2^k\varepsilon).$$

For $|t| < 2^k\varepsilon$ this agrees with previous step, so that we indeed get extension at each step.

To see that $\gamma(t+s) = \gamma(t)\gamma(s)$ as long as $t,s,t+s \in (-2^{k+1}\varepsilon, 2^{k+1}\varepsilon)$, it suffices to check $\gamma'(t) = \gamma(t).x$ for $|t| < 2^{k+1}\varepsilon$, which is straightforward:
$$\gamma'(t) = \tfrac{1}{2}\gamma'(\tfrac{t}{2})\gamma(\tfrac{t}{2}) + \tfrac{1}{2}\gamma(\tfrac{t}{2})\gamma'(\tfrac{t}{2}) = \tfrac{1}{2}\gamma(\tfrac{t}{2})x\,\gamma(\tfrac{t}{2}) + \tfrac{1}{2}\gamma(\tfrac{t}{2})\gamma(\tfrac{t}{2})x \overset{(***)}{=} \gamma(\tfrac{t}{2})^2 x = \gamma(t)x$$

$\Rightarrow$ This provides claimed $\gamma_x : \mathbb{R} \to G$. Uniqueness is clear from above $\blacksquare$

Def 5: Let G be a real Lie group and $\mathfrak{g} = T_1G$. Then, the exponential map
$\boxed{\exp : \mathfrak{g} \to G \text{ is given by } \exp(x) = \gamma_x(1)}$ with γ_x from Prop1

Note: If $\gamma(t)$ satisfied $\gamma'(t) = \gamma(t)\cdot\gamma'(0)$, then for any $c \in \mathbb{R}$ the function $\bar{\gamma}(t) := \gamma(ct)$ satisfies $\bar{\gamma}'(t) = \gamma(ct)\cdot c\gamma'(0) = \bar{\gamma}(t)\cdot c\gamma'(0)$, i.e. by uniqueness:
$$\boxed{\gamma_{cx}(t) = \gamma_x(ct) \quad \forall x \in \mathfrak{g},\ c \in \mathbb{R},\ t \in \mathbb{R}}$$

In particular, we get:

$\boxed{\text{Corollary 2: } \gamma_x(t) = \exp(tx)}$

Examples: 1) For $G = GL_n(\mathbb{R})$ or classical subgroup of $GL_n(\mathbb{R})$, we have $\gamma_x(t) = \exp(tx)$ in the sense of usual exp from Lecture 4.
Hence, for classical gps get the same map as in Lecture 4.

2) For $G = (\mathbb{R}^n, +)$, we have $\gamma_{\vec{x}}(t) = t\cdot\vec{x}$ $\forall \vec{x} \in \mathbb{R}^n$, so that $\exp(\vec{x}) = \vec{x}$.

The key properties of exp are summarized in the following result:

Proposition 2: Let G be a real Lie gp and $\mathfrak{g} = T_1G$.

a) $\exp : \mathfrak{g} \to G$ is a regular map with $\exp(0) = 1$, $d_0\exp = \mathrm{Id}_{\mathfrak{g}}$ and it's a diffeomorphism of nbhd of $0 \in \mathfrak{g}$ onto a nbhd of $1 \in G$.

b) $\exp((t+s)x) = \exp(tx)\cdot\exp(sx)$ $\forall x \in \mathfrak{g}$, $\forall s, t \in \mathbb{R}$

c) $\forall$ morphism of Lie groups $\varphi : G_1 \to G_2$ and $\forall x \in T_1(G_1) =: \mathfrak{g}_1$, we have
$$\exp(\underbrace{\varphi_*(x)}_{\in \mathfrak{g}_2 := T_1(G_2)}) = \varphi(\exp(x))$$

d) For any $g \in G$, $x \in \mathfrak{g}$, we have $g\cdot\exp(x)\cdot g^{-1} = \exp(\mathrm{Ad}_g(x))$

Corollary 3: As an immediate corollary of part a), we see that exp has an inverse map $\log : \underbrace{U}_{\text{some nbhd of } 1 \in G} \to \mathfrak{g}$, called logarithm map

[For classical gps, this coincides with matrix logarithm (Lect 4)

Thus, any basis of $\mathfrak{g}$ gives a local coordinate system near $1 \in G$.

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Proof of Proposition 2

a) Regularity follows from general fact as our ODE depends regularly on x.

$\exp(0) = \gamma_0(1) = 1$ as $\gamma_0(t) = 1 \ \forall t$

$\exp'(0)x = \frac{d}{dt}\big|_{t=0} \exp(tx) \overset{\text{def}}{=} x \Rightarrow \exp'(0) = d_0 \exp = Id_{\mathfrak{g}}$

Thus, by the inverse function theorem, exp is a diffeomorphism near 0.

b) Follows from

$$\exp((t+s)x) = \gamma_x(t+s) = \gamma_x(t)\gamma_x(s) = \exp(tx)\exp(sx)$$

c) Both $\varphi(\exp(tx))$ and $\exp(\varphi_*(tx))$ define 1-parametric subgps of G_2 with the same tangent vector at $1 \in G_2$, equal to $\varphi_*(x)$. By uniqueness they coincide.

d) Follows from c) with $G_1 = G = G_2$ and $\varphi = Ad_g$ ∎

Lemma 2: Let G_1, G_2 be real Lie gps and G_1 - connected.
Then any Lie gp homomorphism $\varphi: G_1 \to G_2$ is uniquely determined by $\varphi_*: \mathfrak{g}_1 \to \mathfrak{g}_2$

By Prop 2(c), φ_* determines φ on a nbhd of $1 \in G_1$.
But by [Lect 3, Prop 1], any open nbhd of connected G_1 generates G_1
$\Rightarrow \varphi$ is uniquely determined ∎

! In fact, all the above results also hold for $\mathbb{C}$ Lie gps (just need corresponding version of ODE), so we have

$$\exp: \mathfrak{g} \to G \quad \text{with all properties above.}$$