Lecture #6

- Last time:
  - vector bundles (clutching f-s)
  - tangent bundle & vector fields
  - 1-parametric subgroups $\gamma_x : \mathbb{R} \to G \quad \forall x \in \mathfrak{g} := T_1 G$ (via sol-n of ODE)
  - the exponential map $\exp : \mathfrak{g} \to G$, $x \mapsto \gamma_x(1)$ (noted: $\gamma_x(t) = \exp(tx)$)

- Start by finishing Prop 2, Cor 3, Lemma 2, $\mathbb{C}$-setup from pp. 5–6 of previous lecture notes.

- So far, $\mathfrak{g} = T_1 G$ was viewed only as a vector space, but ultimately we wish to endow it with a Lie algebra structure. To this end, the Lie bracket will naturally arise from the product on G.

  Consider a small nbhd $0 \in \mathcal{U} \subseteq \mathfrak{g}$, s.t. $\log(\exp(x)\exp(y))$ is defined $\forall x, y \in \mathcal{U}$, which gives rise to some smooth map

(expresses product on group G in local chart near 1) $\rightsquigarrow$
$$\boxed{\begin{aligned} \mu : \mathcal{U} \times \mathcal{U} &\longrightarrow \mathfrak{g} \\ (x, y) &\longmapsto \log(\exp(x) \cdot \exp(y)) \end{aligned}}$$
i.e. $\exp(\mu(x,y)) = \exp(x) \cdot \exp(y)$

> **Lemma 1**: The Taylor series for μ is given by
> $$\mu(x,y) = x + y + \lambda(x,y) + \ldots$$
> where $\lambda : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is bilinear skew-symmetric, $\ldots$ denote terms of order ≥ 3

As $\mu(0,0) = 0$, it can be written as
$$\mu(x,y) = \underbrace{l_1(x) + l_2(y)}_{\text{linear terms}} + \underbrace{q_1(x) + q_2(y)}_{\text{quadratic terms}} + \underbrace{\lambda(x,y)}_{\text{bilinear}} + \text{higher order terms.}$$

But $\exp(0) = 1 \Rightarrow \mu(x,0) \equiv x \;\; \forall x \in \mathcal{U} \Rightarrow l_1(x) \equiv x, \; q_1(x) = 0$
$\Rightarrow \mu(0,y) \equiv y \;\; \forall y \in \mathcal{U} \Rightarrow l_2(y) \equiv y, \; q_2(y) = 0$
$\Big\} \Rightarrow \mu(x,y) = x + y + \lambda(x,y) + \ldots$

Also: $\exp(x) \cdot \exp(x) \overset{\text{Prop 2b}}{=} \exp(2x) \Rightarrow \mu(x,x) = 2x \Rightarrow \lambda(x,x) \equiv 0 \Rightarrow \lambda$ - skew-symmetric

Def 1: The map $\underset{2\cdot\lambda(-,-)}{[-,-]} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is called the commutator.

Thus, $\boxed{\exp(x) \cdot \exp(y) = \exp(x + y + \tfrac{1}{2}[x,y] + \ldots)}$ $\leftarrow$ the reason for factor $\frac{1}{2}$ will be clear soon.

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Example: For $G = GL_n(\mathbb{K})$, $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{K})$, we have

$$\exp(x)\cdot\exp(y) = \left(1+x+\frac{x^2}{2!}+\dots\right)\left(1+y+\frac{y^2}{2!}+\dots\right) = 1+x+y+\left(\frac{x^2}{2}+xy+\frac{y^2}{2}\right)+\dots$$
$$= 1+(x+y)+\frac{(x+y)^2}{2}+\frac{xy-yx}{2}+\dots = \exp\left(x+y+\frac{xy-yx}{2}+\dots\right)$$

Hence: $[x,y] = x\cdot y - y\cdot x \quad \forall x,y\in\mathfrak{gl}_n(\mathbb{K})$ — the usual commutator

As an immediate cosepuence of this example, we get:

Corollary 1: If G is a Lie subgp of $GL_n(\mathbb{K})$, then $\mathfrak{g} = T_1G \subseteq \mathfrak{gl}_n(\mathbb{K})$ and the commutator $[\,,\,]: \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ is the restriction of that on $\mathfrak{gl}_n(\mathbb{K})$

Let's list some basic properties of $[\,,\,]$.

Proposition 1:

a) For a Lie gp homomorphism $\varphi: G_1\to G_2$, its differential $\varphi_*: \underset{T_1G_1}{\mathfrak{g}_1} \to \underset{T_1G_2}{\mathfrak{g}_2}$ preserves $[\,,\,]$:

$$\varphi_*(\underbrace{[x,y]}_{\in\mathfrak{g}_1}) = [\underbrace{\varphi_*(x), \varphi_*(y)}_{\in\mathfrak{g}_2}] \quad \forall x,y\in\mathfrak{g}_1$$

b) The adjoint action $G \overset{Ad}{\curvearrowright} \mathfrak{g} = T_1G$ preserves $[\,,\,]$:

$$Ad\,g([x,y]) = [Ad\,g(x), Ad\,g(y)] \quad \forall x,y\in\mathfrak{g}$$

c) $\exp(x)\exp(y)\exp(-x)\exp(-y) = \exp([x,y] + \underbrace{\dots}_{\text{higher order terms}})$

d) G-commutative $\Rightarrow [x,y]=0 \ \forall x,y\in\mathfrak{g}$.

▶ a) Follows from the def-n of $[\,,\,]$ and Prop 2c) of Lecture 5

b) Special case of a) with $G_1 = G = G_2$ and $\varphi = Ad_g$

c) $\left.\begin{array}{l}\exp(x)\cdot\exp(y) = \exp(x+y+\frac{1}{2}[x,y]+\dots)\\ \exp(-x)\cdot\exp(-y) = \exp(-x-y+\frac{1}{2}[x,y]+\dots)\end{array}\right\} \Rightarrow \begin{array}{l}\exp(x)\exp(y)\exp(-x)\exp(-y)\\ \quad \| \leftarrow \text{as } [x+y,-x-y]=0\\ \exp([x,y]+\dots)\end{array}$

d) Follows immediately from c) ■

Given a Lie gp G, we have a natural adjoint action $G \curvearrowright \mathfrak{g}$, that is

$$Ad: G \to GL(\mathfrak{g}) \ \text{ — Lie gp homomorphism}$$

Def 2: Define $ad := Ad_* = d_1 Ad: \mathfrak{g}\to\mathfrak{gl}(\mathfrak{g})$ — "adjoint action"

First, we note that ad preserves $[\,,\,]$ by Prop 1 a).

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The following are key properties of ad:

Lemma 2:

a) $\operatorname{ad} x(y) = [x, y]$

b) $\operatorname{Ad}(\exp(x)) = \exp(\operatorname{ad} x) : \mathfrak{g} \to \mathfrak{g}$

▷ a) Let's decode the definition of Ad:

$$\operatorname{Ad}_g(y) = \frac{d}{dt}\Big|_{t=0} \left(g \exp(ty) g^{-1}\right)$$

$$\Downarrow$$

$$\operatorname{ad} x(y) = \frac{d}{ds}\frac{d}{dt}\Big|_{s=t=0} \Big(\underbrace{\exp(sx)\exp(ty)\exp(-sx)}_{\exp(ts[x,y]+\dots)\exp(ty) \text{ by Prop 1c)}}\Big) = [x, y]$$

b) Follows from [Prop 2c) of Lecture 5] applied to $G \xrightarrow{\operatorname{Ad}} GL(\mathfrak{g})$.

Alternatively, could argue that both $\exp(t \operatorname{ad} x)$ & $\operatorname{Ad}_{\exp(tx)}$ satisfy same ODE ■

Theorem 1: For any real/complex Lie group G, $(\mathfrak{g}, [\,,\,])$ is a Lie algebra over $\mathbb{R}/\mathbb{C}$

▷ As $[\,,\,]$ is skew-symmetric, we just need to check Jacobi identity.

But $\operatorname{ad}: \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ preserves commutators and we know $[A,B] = AB - BA$ $\forall A, B \in \mathfrak{gl}(\mathfrak{g})$

So: $\operatorname{ad}([x,y]) = \operatorname{ad} x \cdot \operatorname{ad} y - \operatorname{ad} y \cdot \operatorname{ad} x$

$$\Downarrow$$

$$\boxed{[[x,y],z] = [x,[y,z]] - [y,[x,z]] \quad \forall x, y, z \in \mathfrak{g}} \iff \text{Jacobi}$$

■

This finally establishes the long-promised goal

$$\boxed{\text{Lie groups } G \rightsquigarrow \text{Lie algebras } \mathfrak{g} = \operatorname{Lie}(G)}$$

Moreover, as follows from Prop 1a), we have

$$\boxed{\begin{array}{ccc} \operatorname{Hom}_{\text{Lie gps}}(G_1, G_2) & \longrightarrow & \operatorname{Hom}_{\text{Lie alg}}(\mathfrak{g}_1, \mathfrak{g}_2) \\ \varphi & \longmapsto & \varphi_* \end{array}}$$

a linear map of Lie algebras $\mathfrak{g}_1 \to \mathfrak{g}_2$ is homomorphism if it preserves Lie bracket

and this map is injective if G_1 – connected by [Lemma 2 of Lecture 5]

UPSHOT: Have a functor $\operatorname{Lie}: \left\{\begin{array}{c}\text{category of}\\ \text{Lie gps}\end{array}\right\} \to \left\{\begin{array}{c}\text{category of}\\ \text{Lie algebras}\end{array}\right\}$

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Def 3: For any Lie algebra $\mathfrak{g}$ over any field $\mathbb{K}$:

a) a $\mathbb{K}$-subspace $\mathfrak{h} \subseteq \mathfrak{g}$ is called a *Lie subalgebra* if $[x,y] \in \mathfrak{h}$ $\forall x,y \in \mathfrak{h}$

b) a $\mathbb{K}$-subspace $\mathfrak{h} \subseteq \mathfrak{g}$ is called an *ideal* if $[x,y] \in \mathfrak{h}$ $\forall x \in \mathfrak{g}, y \in \mathfrak{h}$.

Proposition 2: Let G be a real/cpx Lie gp with Lie algebra $\mathfrak{g}$.

a) If H is a Lie subgroup in G, then $\mathfrak{h} = T_1 H$ is a Lie subalgebra in $\mathfrak{g}$.

b) If H is a normal closed Lie subgp in G, then $\mathfrak{h} = T_1 H$ is an ideal in $\mathfrak{g}$.

► a) Follows immediately from $\exp(tx) \in H$ $\forall t \in \mathbb{R}$ for $x \in T_1 H$, combined with Prop 1c), see the proof of Lemma 2a) (i.e. look at $\lim\limits_{s,t\to 0} \frac{\log(\exp(sx)\cdot\exp(ty)\exp(-sx)\exp(-ty))}{s\cdot t}$)

b) If $x \in \mathfrak{g}, y \in T_1 H$, then H being normal, we have $\exp(x)\cdot\exp(y)\cdot\exp(-x) \in H$, hence result follows. ◻

Exercise: Show that if H is a closed Lie subgp of G, both H, G are connected, and $T_1 H$ is an ideal in $\mathfrak{g}$, then H is normal

Let us now state 3 *fundamental theorems* of Lie theory

1st Fundamental Thm: For any real/complex Lie group G, there is a bijection between connected Lie subgroups $H \subseteq G$ and Lie subalgebras $\mathfrak{h} \subseteq \mathfrak{g}$, given by $H \mapsto T_1 H$

2nd Fundamental Thm: If G_1, G_2 are real/complex Lie gps and G_1 is simply-connected (+ connected) then the map $\mathrm{Hom}_{\text{Lie gps}}(G_1, G_2) \to \mathrm{Hom}_{\text{Lie alg.}}(\mathrm{Lie}(G_1), \mathrm{Lie}(G_2))$, $\varphi \mapsto \varphi_*$, is bijective

3d Fundamental Thm: Any finite-dimensional real/complex Lie algebra is the Lie algebra of a real/complex Lie group.

We shall skip their proofs for now on (and maybe will return to in the end of the course, if time remains).

Corollary 2: The categories of {finite-dimensional Lie algebras} and {connected, simply-connected Lie groups} are equivalent.