Lecture #7

- Last time
  - For any Lie group G, endowed T_1G with a Lie algebra structure (for $G=GL_n(\mathbb{K})$, recovered usual commutator $[A,B]=AB-BA$ on $gl_n(\mathbb{K})$) getting a functor $\left\{\begin{matrix}\text{category of}\\ \text{Lie gps}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{category}\\ \text{of Lie alg-s}\end{matrix}\right\}$
  - Stated three fundamental theorems of Lie theory, implying an equivalence $\left\{\begin{matrix}\text{category of connected}\\ \text{simply-connected Lie gps}\end{matrix}\right\} \simeq \left\{\begin{matrix}\text{finite-dimensional}\\ \text{Lie algebras (over } \mathbb{R} \text{ or } \mathbb{C})\end{matrix}\right\}$

- Henceforth, we shall mostly focus on the Lie algebras alone and also their representations.

Def 1: a) A *representation* of a Lie algebra $\mathfrak{g}$ over a field $\mathbb{K}$ is a vector space V over $\mathbb{K}$ together with a homomorphism of Lie algebras $\rho:\mathfrak{g}\longrightarrow gl(V)$, i.e. ρ is $\mathbb{K}$-linear and $\boxed{\rho([x,y])=[\rho(x),\rho(y)]=\rho(x)\rho(y)-\rho(y)\rho(x) \quad \forall x,y\in\mathfrak{g}}$

b) Given two representations (V,ρ_V) & (W,ρ_W) of $\mathfrak{g}$, a *homomorphism* from the former to the latter is a linear map commuting with $\mathfrak{g}$-action, i.e.

$$\alpha: V\longrightarrow W \text{ linear s.t. } \boxed{\alpha\circ\rho_V(x)=\rho_W(x)\circ\alpha \quad \forall x\in\mathfrak{g}}$$

Terminology:
- representations of $\mathfrak{g}$ are also called *$\mathfrak{g}$-modules*
- homom. of repr-s are also called *intertwining operators*

Conventions: for $\mathbb{K}=\mathbb{R}$ it's also common to consider complex V (not just real)
this shall actually simplify the theory

As a Corollary of the 2nd Fund. Thm of Lie theory, get:

Proposition 1: Let G be a real/cpx Lie group and $\mathfrak{g}=\mathrm{Lie}(G)$.

a) Every fin. dim. rep. $\rho: G\longrightarrow GL(V)$ defines a Lie algebra repr-n $\rho_*:\mathfrak{g}\longrightarrow gl(V)$ and any morphism of G-representations is also a morphism of $\mathfrak{g}$-modules

b) If G is connected, simply-connected, then $\rho\mapsto\rho_*$ from a) gives an equivalence of categories b/w the corresponding categories of finite dimensional repr-s. In particular, every f. dim. $\mathfrak{g}$-module can be uniquely exponentiated to a representation of G

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Example: Besides for the trivial repr-n, every Lie algebra $\mathfrak{g}$ admits an adjoint representation $\mathfrak{g} \overset{ad}{\curvearrowright} \mathfrak{g}$ given by $ad(x): y \mapsto [x,y]$.
Indeed, as we saw before, ad-representation $\Leftrightarrow$ Jacobi for $[\,,\,]$.

As always, it's important to recall basic procedures to construct $\mathfrak{g}$-modules

Def 2: A subspace W of a $\mathfrak{g}$-module (V, ρ_V) is called a subrepresentation if it is $\mathfrak{g}$-invariant, i.e. $\rho_V(x)W \subseteq W$ $\forall x \in \mathfrak{g}$.

- Similar definition applies to G-modules. In either case, the quotient space V/W carries a natural structure of $\mathfrak{g}$-module (resp G-module), called the quotient representation
- Given a $\mathfrak{g}$-module V, the dual space V^* is endowed with a $\mathfrak{g}$-module structure:

$$\boxed{\rho_{V^*}(x) := -\rho_V(x)^*} \leftarrow \text{dual representation}$$

- Given two module V, W of $\mathfrak{g}$, we can equip $V \oplus W$ with a $\mathfrak{g}$-module structure:

$$\boxed{\rho_{V\oplus W}(x) := \rho_V(x) \oplus \rho_W(x)} \leftarrow \text{direct sum of representations}$$

- — // —, we can equip $V \otimes W$ with a $\mathfrak{g}$-module structure:

$$\boxed{\rho_{V\otimes W}(x) := \rho_V(x) \otimes \mathrm{id}_W + \mathrm{id}_V \otimes \rho_W(x)} \leftarrow \text{tensor product of represent-s}$$

Exercise 1: a) Verify the above f-las define $\mathfrak{g}$-actions
b)* For $\mathfrak{g} = \mathrm{Lie}(G)$, derive these formulas from the corresponding constructions for G-modules
c) Identifying $V^* \otimes W \simeq \mathrm{Hom}(V, W)$ as vector spaces, construct a natural action of $\mathfrak{g}$ on $\mathrm{Hom}(V, W)$, $\forall \mathfrak{g}$-modules V, W.
d) Verify $\mathrm{Hom}(V, W)^{\mathfrak{g}} = \mathrm{Hom}_{\mathfrak{g}}(V, W)$ [see Notation below for LHS]

In particular, above constructions endow $S^n V$, $\Lambda^n V$, $V^{\otimes n} \otimes (V^*)^m$ with canonical $\mathfrak{g}$-module structures.

Notation: For any $\mathfrak{g}$-module V, the $\mathfrak{g}$-invariants are $V^{\mathfrak{g}} := \{v \in V \mid \rho_V(x)v = 0 \ \forall x \in \mathfrak{g}\}$

Major problem in representation theory is to classify all representations of a Lie group or a Lie algebra. To this end, we start with building blocks:

Def 3: a) A representation $V \neq 0$ of G or $\mathfrak{g}$ is *irreducible* if it has no subrepresentation other than V or 0.

b) A representation $V \neq 0$ of G or $\mathfrak{g}$ is *indecomposable* if it does not admit decomposition into direct sum of nonzero modules, i.e. $V \underset{\mathfrak{g}\text{-mod}}{\simeq} W_1 \oplus W_2 \Rightarrow W_1$ or W_2 are zero.

Clearly, every finite dimensional repr-n is isomorphic to $\oplus$ indecomposable, but not necessarily into $\oplus$ irreducibles (give a counter-example)

Def 4: A repr-n V is called *completely reducible* (a.k.a. *semisimple*) if it is $\simeq \oplus$ irreducibles, i.e. $V \simeq \oplus n_i V_i$ (n_i are called *multiplicities*)
V_i - pairwise nonisomorphic irreducible modules

Thus, the major questions are:

(1) classify all irreducible repr-s of G or $\mathfrak{g}$

(2) for a given semisimple repr-n V of G or $\mathfrak{g}$, find all the multiplicities

(3) for which G or $\mathfrak{g}$, all fin. dim. repr-s are completely reducible?

The following result is almost obvious (easy exercise to do at home):

Lemma 1: Let V be a fin. dimensional representation of G or $\mathfrak{g}$, and let $A: V \to V$ be a homomorphism of $\mathfrak{g}$-repr-s. Then $V = \bigoplus_\lambda V(\lambda)$ is a decomposition of representations
$V(\lambda)$ ← generalized eigenspace of A with eigenvalue λ.

Exercise 2: Verify that $V, S^m V, \Lambda^m V$ are irreducible representations of the group $GL(V)$ or a Lie algebra $\mathfrak{gl}(V)$. Deduce that $V \otimes V$ is completely reducible.

As for finite groups, we shall start with the basic Schur's lemma

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Lemma 2 (Schur Lemma): Let V, W be irreducible fin. dim. $\mathbb{C}$-representations of the group G or a Lie algebra $\mathfrak{g}$. Then:

a) if $V \not\simeq W$, we have $\mathrm{Hom}_{G \text{ or } \mathfrak{g}}(V, W) = 0$, i.e. no nonzero homomorphisms

b) if $V \simeq W$, then $\mathrm{Hom}_{G \text{ or } \mathfrak{g}}(V, V) = \mathbb{C} \cdot \mathrm{Id}$

▸ Let $A: V \longrightarrow W$ be a nonzero homomorphism of repr-s.

Then $\mathrm{Ker}\, A$ is a proper submodule of $V \underset{V\text{-irreduc}}{\Longrightarrow} \mathrm{Ker}\, A = 0$

$\mathrm{Im}\, A$ is a nonzero submodule of $W \underset{W\text{-irred.}}{\Longrightarrow} \mathrm{Im}\, A = W$

$\Big\} \Rightarrow A: V \xrightarrow{\sim} W$.

This proves part a). For part b), let $\lambda \in \mathbb{C}$ be an eigenvalue of A, then $A - \lambda \cdot \mathrm{Id}$ is still a homomorphism, but has nontrivial kernel $\Rightarrow$ $\Rightarrow A - \lambda \mathrm{Id} = 0. \Rightarrow A = \lambda \mathrm{Id}_V$. This proves part b) ∎

Note: As part b) may fail over $\mathbb{R}$ (give counterexample!), we will mostly consider $\mathbb{C}$-representations

As the classical corollaries of Schur's lemma, we get:

Corollary 1: The center of G or $\mathfrak{g}$ acts on any irreducible repr-n via scalars. In particular, every irreducible repr-n of abelian G or $\mathfrak{g}$ is 1-dim.

Corollary 2: Let $\{V_i\}$ be irreducible pairwise non-isomorphic complex representations of G or $\mathfrak{g}$, and $V = \oplus n_i V_i$, $W = \oplus m_i V_i$. Then:

a) there is a vector space isomorphism $\mathrm{Hom}_{G \text{ or } \mathfrak{g}}(V, W) \simeq \bigoplus_i \mathrm{Mat}_{m_i \times n_i}(\mathbb{C})$

b) for $W = V$, the isomorphism in a) is an algebra isomorphism

We conclude today's lecture with the discussion of $\mathfrak{sl}_2(\mathbb{C})$ repr. theory.
Down-to-earth, $\mathfrak{sl}_2(\mathbb{C})$ has a basis

$$e=\begin{pmatrix}0&1\\0&0\end{pmatrix},\quad h=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad f=\begin{pmatrix}0&0\\1&0\end{pmatrix}$$

with a commutator

$$[e,f]=h,\quad [h,e]=2e,\quad [h,f]=-2f.$$

Exercise 3: Prove these are defining relations for $\mathfrak{sl}_2$.

Note that $\mathrm{Mat}_{2\times 2}(\mathbb{C}) \curvearrowright \mathbb{C}^2$ actually gives rise to the action (check!)

$$\boxed{\mathfrak{sl}_2 \curvearrowright \mathbb{C}[x,y]=:V \quad \text{via} \quad e\mapsto x\partial_y,\ h\mapsto x\partial_x-y\partial_y,\ f=y\partial_x}$$

each operator preserves degrees and so $V\simeq \bigoplus_{n\geq 0} V_n$ is an $\mathfrak{sl}_2$-decomposition, where $V_n=\{\text{degree } n \text{ polynomials in } x,y\}$, $\dim V_n=n+1$.

Down-to-earth, V_n has a basis $\{\underbrace{x^k y^{n-k}}_{=:v_{k,n-k}} \mid 0\leq k\leq n\}$ and action is

$$\begin{cases} e(v_{k,n-k})=(n-k)v_{k+1,n-(k+1)} \\ h(v_{k,n-k})=(2k-n)v_{k,n-k} \\ f(v_{k,n-k})=k\, v_{k-1,n-(k-1)}. \end{cases}$$

The following is the key result on $\mathfrak{sl}_2$-theory (which will be later used to treat most general simple Lie algebras):

Theorem 1:
a) V_n are irreducible $\mathfrak{sl}_2$-modules

b) If $V\neq 0$ is a fin. dim. $\mathfrak{sl}_2$-module, then e & f are nilpotent on V. Therefore, $U:=\mathrm{Ker}(e)\neq 0$. Finally, h preserves U, acting diagonally with $\mathbb{Z}_{\geq 0}$ coefficients

c) Any irreducible $\mathfrak{sl}_2$-module is $\simeq V_n$ for some n

d) Any f.dim. $\mathfrak{sl}_2$-module is completely reducible

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Let us first discuss some corollaries of the theorem above.

Corollary 3 (Jacobson Morozov thm for $\mathfrak{gl}(V)$): Given a fin. dimensional vector space $\tilde{V}$ over $\mathbb{C}$ and a nilpotent operator $E: \tilde{V} \to \tilde{V}$, there is an $\mathfrak{sl}_2$-action on $\tilde{V}$ $\mathfrak{sl}_2 \overset{\rho}{\curvearrowright} \tilde{V}$ with $\rho(e) = E$

✎ Bring E to a Jordan normal form with diagonal blocks $n_i \times n_i$ (and zero on diagonal as E-nilpotent). As in V_n, we have $e \mapsto$ single $n \times n$ Jordan block, deduce $\tilde{V} \underset{\mathfrak{sl}_2\text{-action}}{\simeq} \bigoplus V_{n_i}$ ⧄

Corollary 4 (Clebsch-Gordan formula): We have $V_n \otimes V_m \simeq \bigoplus\limits_{i=0}^{\min(n,m)} V_{2i+|n-m|}$

The simplest proof uses the notion of characters.

Def: For an $\mathfrak{sl}_2$-module V (of finite dim.), define its *character*

$$\boxed{\chi_V(z) := \mathrm{tr}_V(z^h) = \sum_m \dim \underbrace{V(m)}_{\text{eigenspace of } h_V \text{ of eigenvalue } \lambda} \cdot z^m}$$

Exercise 4: a) Verify $\chi_{V \oplus W}(z) = \chi_V(z) + \chi_W(z)$

$\chi_{V \otimes W}(z) = \chi_V(z) \cdot \chi_W(z)$

b) Verify $\chi_{V_n}(z) = \frac{z^{n+1} - z^{-n-1}}{z - z^{-1}}$ and check they are lin. indep.

c) Prove Corollary 4 by comparing characters of both sides.

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Proof of Theorem 1 a)-c)

a) As $h|_{V_n}$ is diagonal with distinct eigenvalues, any h-stable subspace W of V_n is spanned by a collection of $v_{k,n-k}$.
Applying operators e & f, we see that all $v_{\ell,n-\ell}$ are in $W \Rightarrow W = V_n$

b) $V = \underset{\lambda\in\mathbb{C}}{\oplus} \underbrace{V(\lambda)}_{\text{generalized } \lambda\text{-eigenspace of } h\text{-action on } V}$

Easy: $\left.\begin{array}{l} eV(\lambda) \subseteq V(\lambda+2) \\ fV(\lambda) \subseteq V(\lambda-2) \end{array}\right\} \xRightarrow{V\text{-fin. dim}}$ e and f act nilpotently on V

So: $\mathcal{U} := \mathrm{Ker}(e)$ is non-empty.
$\forall v \in \mathcal{U}$: $e(hv) = (he - 2e)v = 0 \Rightarrow \mathcal{U}$ is h-invariant

Exercise: a) Check that $ef^m v = f^{m-1}\cdot m(h-m+1)v \quad \forall v \in V$
b) Deduce $e^m f^m v = e^{m-1} f^{m-1} \cdot m(h-m+1)v = \ldots = m!\, h(h-1)\ldots(h-m+1)v$
c) As $f^m \equiv 0$ for $m > \dim V$, get $h(h-1)\ldots(h-m+1) \equiv 0$ on $\mathcal{U}$.
Deduce part b)

c) As $\mathcal{U}$ is h-invariant, there is $v \in \mathcal{U}\setminus\{0\}$ s.t. $hv = \lambda v$.
Consider a sequence
$v_0 = v,\ v_1 = fv,\ v_2 = f^2 v, \ldots$
Clearly: $fv_m = v_{m+1}$ and $hv_m = (\lambda - 2m)v_m$
By above: $ev_m = m(\lambda - m + 1)v_{m-1}$.
Note that $\{v_m \neq 0\}$ are lin. indep as they have different h-eigenvalues
But V-fin. dim $\Rightarrow \exists n$ s.t. $v_{n+1} = 0$ but $v_{\leq n} \neq 0$. Then:
$0 = ev_{n+1} = (n+1)(\lambda + 1 - (n+1))v_n \Rightarrow \lambda = n$

Moreover, $\left.\begin{array}{l} V\text{-irreducible} \\ \cup\!\!\mid \\ \mathrm{span}\{v_0, \ldots, v_n\}\text{-submodule} \end{array}\right. \Rightarrow V = \mathrm{span}\{v_0, \ldots, v_n\}$

Now it's easy to see $V \simeq V_n$ (check!)

We will prove part d) next time!