Lecture #8

- Last time:
  - Category of fin. dim. $\mathfrak{g}$-modules
  - quotient, dual, $\oplus$, $\otimes$ of $\mathfrak{g}$-modules
  - Schur Lemma
  - Main Theorem for $\mathfrak{sl}_2$ (proved only the first two parts)
  - characters of $\mathfrak{sl}_2$-modules (application to Clebsch-Gordan formula)

- We shall start today's class by finishing the proof of this Main Thm for $\mathfrak{sl}_2$.

c) — see the end of notes for Lecture #7.

d) The proof of the complete reducibility of any fin. dim. $\mathfrak{sl}_2$-module V requires a new technique/idea, namely, consider the following operator on V:

$$\boxed{\text{Casimir operator} \quad C_V := 2\rho_V(f)\rho_V(e) + \frac{\rho_V(h)^2}{2} + \rho_V(h) = \rho_V(f)\rho_V(e) + \rho_V(e)\rho_V(f) + \frac{\rho_V(h)^2}{2}}$$

Easy Check: a) $C_V : V \to V$ is a homomorphism of $\mathfrak{sl}_2$-modules, i.e.

$$\boxed{[C_V, \rho_V(x)] = 0 \quad \text{for } x = e, h, f.}$$

b) On irreducible modules V_n (see part a)), it acts by scalar (Schur Lemma) which can be computed from the action on $x^n y^0 \in V_n$, so that

$$\boxed{C_{V_n} = \frac{n^2}{2} + n = \frac{n(n+2)}{2}}$$

Now, given an $\mathfrak{sl}_2$-module (V, ρ), let's first split it as $\oplus$ indecomposables. Henceforth, we may assume V is indecomposable. By Lemma 1 from Lect 7:
$V = \bigoplus_\lambda \underbrace{V(\lambda)}_{\substack{\text{generalized} \\ \text{eigenspace of } C_V}}$ as $\mathfrak{sl}_2$-modules, hence, $V = V(\lambda)$ for some λ (as V is indecomposable)

So: The Casimir operator acts on V with a single eigenvalue λ.

Now, consider any Jordan-Hölder filtration of V, which by part c) means a filtration of $\mathfrak{sl}_2$-modules $\quad 0 = W_0 \subseteq W_1 \subseteq W_2 \subseteq \ldots \subseteq W_{N-1} \subseteq W_N = V$ with $W_k / W_{k-1} \simeq V_{n_k}$ as $\mathfrak{sl}_2$-modules. But then $\lambda = \frac{n_k(n_k+2)}{2}$ $\forall k \Rightarrow n_1 = n_2 = \ldots = n_N =: n$.
In particular, $\dim V = N(n+1)$, as well as $\dim V(\ell) = \begin{cases} N & \text{if } \ell \in \{n, n-2, \ldots, -n+2, -n\} \\ 0 & \text{, else} \end{cases}$

Pick a basis $v^{(1)}, \ldots, v^{(N)}$ of $V(n)$, the "highest weight component" w.r.t. $\rho_V(h)$.

Lecture #8

(Continuation)

For each $1 \le k \le N$, the vectors $\{v_0^{(k)} := v^{(k)}, v_1^{(k)} := f v^{(k)}, \ldots, v_n^{(k)} := f^n v^{(k)}\}$ will span an $\mathfrak{sl}_2$-submodule of V, to be denoted by U_k, s.t. $U_k \simeq V_n$.

Easy Check (Exercise): The natural inclusions $U_k \hookrightarrow V$ give rise to an $\mathfrak{sl}_2$-module embedding $U_1 \oplus U_2 \oplus \ldots \oplus U_N \overset{\iota}{\hookrightarrow} V$ (i.e. the sum is actually direct!)

As $\dim V = N(n+1) = \dim(U_1 \oplus \ldots \oplus U_N)$, we conclude ι is isomorphism! ∎

- For the rest of today, we shall study universal enveloping algebras. Before we give a definition, let's recall a notion of a tensor algebra of a vector space V (over a field $\mathbb{k}$):

$$T(V) := \mathbb{k} \oplus V \oplus V \otimes V \oplus V \otimes V \otimes V \oplus \ldots = \bigoplus_{n \ge 0} V^{\otimes n}$$

with the multiplication $V^{\otimes n} \times V^{\otimes m} \longrightarrow V^{\otimes (n+m)}$

$$(v_1 \otimes \ldots \otimes v_n), (w_1 \otimes \ldots \otimes w_m) \mapsto v_1 \otimes \ldots \otimes v_n \otimes w_1 \otimes \ldots \otimes w_m$$

The tensor algebra $T(V)$ is $\mathbb{Z}_{\ge 0}$-graded, with degree n component being $V^{\otimes n}$.

Down-to-earth, picking a basis $\{v_i\}$ of V, $T(V)$ is identified with the free algebra in $\{v_i\}$.

Def 1: For a Lie algebra $\mathfrak{g}$ over a field $\mathbb{k}$, the universal enveloping algebra of $\mathfrak{g}$, denoted by $U(\mathfrak{g})$, is defined as the quotient of the tensor algebra $T(\mathfrak{g})$ by the ideal generated by $\{x \cdot y - y \cdot x - [x,y] \mid x, y \in \mathfrak{g}\}$

Let $\boxed{\rho: \mathfrak{g} \longrightarrow U(\mathfrak{g})}$ be the natural map, i.e. composition of $\mathfrak{g} \to T(\mathfrak{g}) \twoheadrightarrow U(\mathfrak{g})$

The following is straightforward:

Lemma 1: For any associative algebra A over $\mathbb{k}$, the map

$$\mathrm{Hom}_{\mathrm{ass.alg}}(U(\mathfrak{g}), A) \longrightarrow \mathrm{Hom}_{\mathrm{Lie\,alg}}(\mathfrak{g}, A), \quad \varphi \mapsto \varphi \circ \rho$$

is a bijection (where we view A as a Lie algebra via $[a,b] := a \cdot b - b \cdot a$)

This universal property of $U(\mathfrak{g})$ characterizes it uniquely and explains the terminology.

Lecture #8

As an immediate corollary, we get:

Corollary 1: Any $\mathfrak{g}$-module has a natural structure of a $U(\mathfrak{g})$-module, and vice versa each $U(\mathfrak{g})$-module has a natural structure of $\mathfrak{g}$-mod

Thus, understanding a structure of $U(\mathfrak{g})$ is useful in the study of $\mathfrak{g}$-modules.

Example: The Casimir operator C_V we introduced above in the proof of part d) of Main Thm is just coming from an action of an element $C := e\cdot f + f\cdot e + \frac{1}{2} h\cdot h \in U(\mathfrak{sl}_2)$. In particular, C_V being a homomorph. of $\mathfrak{sl}_2$-modules is just the "shadow" of the following result a).

Exercise: a) C - central element of $U(\mathfrak{sl}_2)$

b)* The center of $U(\mathfrak{sl}_2)$ is just $\mathbb{k}[C]$.

Down-to-earth, $U(\mathfrak{g})$ can be described as the quotient of the free algebra $\mathbb{k}\langle\{x_i\}\rangle$ by the rel-s $x_i x_j - x_j x_i - \sum_k c_{ij}^k x_k = 0$, where $\{x_i\}$ - basis of $\mathfrak{g}$, and $\{c_{ij}^k\}$ are structure constants of the Lie bracket on $\mathfrak{g}$, i.e. $[x_i, x_j] = \sum_k c_{ij}^k x_k$.

Simplest Example: $\mathfrak{g}$ - abelian, i.e. $[x,y]=0 \ \forall x,y \in \mathfrak{g}$

Then, $U(\mathfrak{g}) \simeq S(\mathfrak{g}) \simeq \mathbb{k}[\{x_i\}]$

Observation: Recall that any Lie algebra $\mathfrak{g}$ acts on itself via $\mathrm{ad}: \mathfrak{g} \curvearrowright \mathfrak{g}$. This gives rise to a natural action $\mathfrak{g} \curvearrowright T(\mathfrak{g})$ via derivations and we claim that it descends to $\mathfrak{g} \curvearrowright U(\mathfrak{g})$. To this end, we shall verify that $\mathrm{Ker}(T(\mathfrak{g}) \twoheadrightarrow U(\mathfrak{g}))$ is ad-invariant

key verification

$$\mathrm{ad}\,x(y\cdot z - z\cdot y - [y,z]) = [x,y]\cdot z + y\cdot[x,z] - [x,z]\cdot y - z\cdot[x,y]$$
$$- [x,[y,z]] = \underbrace{\left([x,y]\cdot z - z\cdot[x,y] - [[x,y],z]\right)}_{\text{of the form } a\cdot b - b\cdot a - [a,b]} + \underbrace{\left(y\cdot[x,z] - [x,z]\cdot y - [y,[x,z]]\right)}_{-//-}$$
$$+ \underbrace{[[x,y],z] + [y,[x,z]] - [x,[y,z]]}_{=0 \text{ by Jacobi}}$$

So: $\mathfrak{g} \overset{\mathrm{ad}}{\curvearrowright} U(\mathfrak{g})$ via derivations

As $\mathrm{ad}(x)(y) = [x,y] \overset{\text{in } U(\mathfrak{g})}{=} x\cdot y - y\cdot x \ \forall y \in \mathfrak{g}$, we actually see that $\mathrm{ad}(x)$ is an inner derivation of $U(\mathfrak{g})$ given by $y \mapsto x\cdot y - y\cdot x$

Lecture #8

We can summarize the above discussion by

Lemma 2: The adjoint action of $\overset{ad}{\curvearrowright} \mathfrak{g}$ gives rise to an action of $\mathfrak{g} \overset{ad}{\curvearrowright} U\mathfrak{g}$ via inner derivations.

As an immediate Corollary, we get:

Corollary 2: The center $ZU(\mathfrak{g})$ is isomorphic to $\mathfrak{g}$-invariants of $U\mathfrak{g}$, i.e.

$$ZU(\mathfrak{g}) \simeq U(\mathfrak{g})^{\mathfrak{g}}$$

see Hwk 4, #1c).

To state one of the key structural results on $U(\mathfrak{g})$, the PBW theorem, let us first recall a general notion of filtered algebras and associated graded.

Def 2: a) A $\mathbb{Z}_{\geq 0}$-filtered associative algebra is an assoc. algebra A with a filtration $0 \subseteq F_0 A \subseteq F_1 A \subseteq F_2 A \subseteq \ldots$ by vector subspaces, s.t.

- $1 \in F_0 A$
- $\bigcup_{n\geq 0} F_n A = A$ ("exhaustive filtration")
- w.r.t. product on A, we have $F_k A \cdot F_\ell A \subseteq F_{k+\ell} A$

b) The associated graded algebra of A with $F_\cdot A$ as in part a) is the algebra, denoted by $gr(A)$ or $gr_{F_\cdot} A$, which as v.space is:

$$\boxed{gr(A) = \bigoplus_{k\geq 0} \underbrace{F_k A / F_{k-1} A}_{=: gr_k A} = \bigoplus_{k\geq 0} gr_k(A)}$$

while the product is defined as follows

given $\bar{x} \in gr_k A$, $\bar{y} \in gr_\ell A$ pick their (non-unique) "lifts" $x \in F_k A$, $y \in F_\ell A$ and define $\underset{\text{product in } gr(A)}{\bar{x} \cdot \bar{y}} := \underset{\text{product in } A}{\overline{x \cdot y}}$

Exercise: a) Verify the above product is well-defined.

b) Prove that if $gr(A)$ has no zero divisors, then so is A.

Note: If the algebra A was already $\mathbb{Z}_{\geq 0}$-graded, i.e. $A = \bigoplus_{k\geq 0} A_k$ with $A_k \cdot A_\ell \subseteq A_{k+\ell}$, then it has a canonical $\mathbb{Z}_{\geq 0}$-filtration with $F_k A := \bigoplus_{\ell=0}^{k} A_k$ and $gr_{F_\cdot} A \simeq A$ as algebras. For example, tensor algebra $T(V)$ was $\mathbb{Z}_{\geq 0}$-graded, hence, $\mathbb{Z}_{\geq 0}$-filtered.

Lecture # 8

If A is a $\mathbb{Z}_{\geq 0}$-graded algebra and $I \subseteq A$ is a 2-sided ideal which is also $\mathbb{Z}_{\geq 0}$-graded, i.e. $I = \bigoplus_{k\geq 0} A_k \cap I$, then clearly A/I is a $\mathbb{Z}_{\geq 0}$-graded algebra. However, if I is not $\mathbb{Z}_{\geq 0}$-graded, then the quotient algebra A/I acquires only a $\mathbb{Z}_{\geq 0}$-filtration via $\boxed{(A/I)_k := \text{image of } F_k A = \bigoplus_{\ell=0}^{k} A_\ell \text{ under } A \twoheadrightarrow A/I.}$

- Let us now apply the above general setup in case of interest.

$$\boxed{\underbrace{\mathfrak{g}}_{\text{Lie alg}} \rightsquigarrow \underbrace{T(\mathfrak{g})}_{\mathbb{Z}_{\geq 0}\text{-graded}} \text{ - tensor algebra} \rightsquigarrow \underbrace{U(\mathfrak{g})}_{\text{only } \mathbb{Z}_{\geq 0}\text{-filtered}} = T(\mathfrak{g})/I \text{ - universal enveloping alg.}}$$

Explicitly, $F_k U(\mathfrak{g})$ is defined as the image of $\bigoplus_{\ell=0}^{k} \mathfrak{g}^{\otimes \ell} \subset T(\mathfrak{g})$

Note: $\forall x, y \in \mathfrak{g}$, have $x \cdot y - y \cdot x = [x, y] \in F_1 U(\mathfrak{g})$

Easy Check (do it!): for $x \in F_k U(\mathfrak{g})$, $y \in F_\ell U(\mathfrak{g})$, we have $[x, y] \in F_{k+\ell-1} U(\mathfrak{g})$

Thus, the associated graded algebra $\mathrm{gr}.\, U(\mathfrak{g})$ is commutative!
Moreover, as $U(\mathfrak{g})$ is generated by $\mathfrak{g}$ (the degree 1 component: $\mathfrak{g} \subseteq F_1 U(\mathfrak{g})$) so does $\mathrm{gr}.\, U(\mathfrak{g})$. $\Big\} \Rightarrow$

$\Rightarrow$ get an algebra epimorphism $\boxed{S(\mathfrak{g}) \overset{\phi}{\twoheadrightarrow} \mathrm{gr}\, U(\mathfrak{g})}$

Theorem 1 (Poincare - Birkhoff - Witt theorem or PBW theorem for short): ϕ is an isomorphism

Before starting the proof of this fundamental result, let's discuss some of useful corollaries. We start with the classical reformulation of it:

Corollary 3: For any basis $\{x_i\}$ of $\mathfrak{g}$ and any ordering $<$ on it, the ordered monomials $\overset{<}{\prod_i} x_i^{n_i}$ form a basis of $U(\mathfrak{g})$.

Note: The easy part is that the ordered monomials span $U(\mathfrak{g})$. Thus, the core of the PBW thm is actually linear independence of ordered monomials.

Corollary 4: The map $\rho: \mathfrak{g} \to U\mathfrak{g}$ is injective