Lecture #9

- Last time:
  - finished the Main Theorem about fin. dim. $\mathfrak{sl}_2$-modules
    (key ingredients: Casimir Operator, Jordan - Hölder filtration)
  - introduced universal enveloping algebra $U(\mathfrak{g})$ $\left(\Rightarrow \mathfrak{g}\text{-modules} = U(\mathfrak{g})\text{-modules}\right)$
    $\hookrightarrow$ discussed its universal property
  - discussed $\mathfrak{g} \overset{ad}{\curvearrowright} \mathfrak{g} \rightsquigarrow \mathfrak{g} \curvearrowright T\mathfrak{g} \rightsquigarrow \mathfrak{g} \overset{ad}{\curvearrowright} U\mathfrak{g}$ via inner derivations
    [Note: $\mathfrak{g} \curvearrowright T\mathfrak{g}$ is not given by $x \cdot \_ - \_ \cdot x$!)
    $\Downarrow$
    $\boxed{ZU(\mathfrak{g}) \simeq U(\mathfrak{g})^{\mathfrak{g}}}$ ← via adjoint action
  - $\mathbb{Z}_{\geq 0}$-filtered algebras and associated graded
  - Stated two equivalent formulations of the Poincare - Birkhoff - Witt theorem
    1) $S(\mathfrak{g}) \xrightarrow{\sim} \mathrm{gr}\, U(\mathfrak{g})$ is basis-indep. formulation
    2) $\forall$ ordered basis $\{x_1, \ldots, x_n\}$ of $\mathfrak{g}$, ordered monomials $\{x_1^{k_1} \ldots x_n^{k_n}\}$ - basis of $U(\mathfrak{g})$ is basis dependent formulation

  Cor 1: the natural map $\mathfrak{g} \to U(\mathfrak{g})$ is injective

  Cor 2: If $\mathfrak{g}_1, \ldots, \mathfrak{g}_m$ are Lie subalgebras of $\mathfrak{g}$ s.t. $\mathfrak{g} \overset{v.\,space}{=\!=} \bigoplus_{k=1}^{m} \mathfrak{g}_k$, then mult: $U(\mathfrak{g}_1) \otimes \_\_ \otimes U(\mathfrak{g}_m) \to U(\mathfrak{g})$ is an isom. of v. spaces

- Hand out Homeworks 3 + comment on several problems

- Start today's lecture by making a couple of useful comments on Lect 7-8.

  * Verma $\mathfrak{sl}_2$-module

    For any $\lambda \in \mathbb{C}$, consider a vector space M_λ with a basis $\{v_k \mid k \geq 0\}$ and action

    $$\boxed{h(v_k) = (\lambda - 2k)v_k, \quad f(v_k) = (k+1)v_{k+1}, \quad e(v_k) = (\lambda - k + 1)v_{k-1}}$$

    (Clear: $[h, e] = 2e$, $[h, f] = -2f$
    Also: $[e, f]: v_k \mapsto \left((k+1)(\lambda - k) - k(\lambda - k + 1)\right) v_k = (\lambda - 2k)v_k \Rightarrow [e, f] = h$)

    So: Above indeed defines an $\mathfrak{sl}_2$-action on M_λ, called the Verma module

    Using that h is diagonal in the above basis with pairwise distinct eigenvalues we get:

    $$\boxed{\begin{array}{l} 1)\ M_\lambda \text{ is irreducible if } \lambda \notin \mathbb{Z}_{\geq 0} \\ 2)\ \text{If } \lambda = n \in \mathbb{Z}_{\geq 0}, \text{ then } \mathrm{Span}\{v_{n+1}, v_{n+2}, \ldots\} \text{ is a submodule} \simeq M_{-n-2} \end{array}}$$

Lecture #9
(Continuation)

So: For $n \in \mathbb{Z}_{\geq 0}$, we actually have a short exact sequence of $\mathfrak{sl}_2$-modules

$$\boxed{0 \to M_{-n-2} \to M_n \to V_n \to 0}$$

In particular, this "categorifies" the character f-la from Lecture 7:

$$\boxed{\chi_{V_n}(z) = \frac{z^{n+1} - z^{-n-1}}{z - z^{-1}} = \frac{z^n}{1-z^{-2}} - \frac{z^{-n-2}}{1-z^{-2}} = \chi_{M_n}(z) - \chi_{M_{-n-2}}(z)}$$

Rmk: In the end of our course, we'll learn the general Weyl character f-la (and at least the statement of Bernshtein – Gelfand – Gelfand resolution)

* $\mathbb{Z}_2$-symmetry of weights

Since any fin. dim. $\mathfrak{sl}_2$-module is $\simeq \bigoplus_{k=1}^{m} V_{n_k}$, and we know how the elt $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ acts in V_{n_k}, we immediately get

> Corollary: Any fin. dim. $\mathfrak{sl}_2$-module V admits a "weight decomposition"
> $$V = \bigoplus_{n \in \mathbb{Z}} V(n), \quad V(n) = \{v \in V \mid h(v) = n \cdot v\}$$
> and $e^n : V(-n) \xrightarrow{\sim} V(n)$, $f^n : V(n) \xrightarrow{\sim} V(-n)$, hence $\dim V(n) = \dim V(-n)$ $\forall n$

! Rmk: The above fails for ∞-dim $\mathfrak{sl}_2$-modules, see e.g. Verma M_λ.

* A few comments on PBW

1) While $U(\mathfrak{g})$ & $S(\mathfrak{g})$ are ∞-dim, the graded pieces of $S(\mathfrak{g})$ & $\mathrm{gr}\, U(\mathfrak{g})$ are fin. dim!
2) For $\mathfrak{g}$-abelian, i.e. $[x,y]=0$, have $U(\mathfrak{g}) \simeq S(\mathfrak{g})$
3) Easy: if $\{x_1, \ldots, x_n\}$ – basis of $\mathfrak{g}$ $\Rightarrow$ $\{x_1^{k_1} \ldots x_n^{k_n}\}$ span $U(\mathfrak{g})$

Follows by induction, where one shows that $\{x_1^{k_1} \ldots x_n^{k_n} \mid k_1 + \ldots + k_n \leq M\}$ span $F_M U(\mathfrak{g})$

Step of induction: $F_{M+1} U(\mathfrak{g})$ is spanned by $\{x \cdot y \mid x \in \mathfrak{g}, y \in F_M U(\mathfrak{g})\}$, hence, by hypothesis $F_{M+1} U(\mathfrak{g})$ is spanned by $\{x_i \cdot x_1^{k_1} \ldots x_i^{k_i} \ldots x_n^{k_n} \mid k_1 + \ldots + k_n \leq M\}$

But: $x_i \cdot x_1^{k_1} \ldots x_i^{k_i} \ldots x_n^{k_n} = \underbrace{x_1^{k_1} \ldots x_i^{k_i+1} \ldots x_n^{k_n}}_{\text{ordered monomials}} + (\text{l.o.t.})$

$$\underbrace{[x_i, x_1] \cdot x_1^{k_1-1} \ldots x_i^{k_i} \ldots x_n^{k_n} + x_1 \cdot [x_i, x_1] x_1^{k_1-2} \ldots x_n^{k_n} + \ldots + x_1^{k_1} \ldots x_{i-1}^{k_{i-1}-1} [x_i, x_{i-1}] x_i^{k_i} \ldots x_n^{k_n}}_{\in F_M U(\mathfrak{g}) \Rightarrow \text{can apply induction hypothesis}}$$

- New material (continuation of PBW) ← I will not prove PBW thm in class (postpone till hwk OR undegraduate student's talk in the end)

Since $S(\mathfrak{g})$ is a domain (as polynomial rings don't have zero divisors), we get (use [Homework 4, Problem 4a)]):

Corollary 1: $U(\mathfrak{g})$ is a domain

A natural question to ask when looking at PBW isomorphism $\phi: S(\mathfrak{g}) \xrightarrow{\sim} \mathrm{gr}\, U(\mathfrak{g})$:

Q: Can ϕ be upgraded to a linear map $S(\mathfrak{g}) \xrightarrow{\sim} U(\mathfrak{g})$ with some properties?

The partial answer is provided by the so-called symmetrization map. Assume $\mathrm{char}(\mathbb{k})=0$, and define

symmetrization map
$$\sigma: S(\mathfrak{g}) \longrightarrow U(\mathfrak{g})$$
$$x_1 \otimes \ldots \otimes x_n \longmapsto \frac{1}{n!}\sum_{\sigma\in S(n)} x_{\sigma(1)}\cdot\ldots\cdot x_{\sigma(n)}$$

[**Exercise**: Verify that σ intertwines adjoint $\mathfrak{g}$-actions on both sides.

Lemma 1: $\sigma: S(\mathfrak{g}) \to U(\mathfrak{g})$ is an isomorphism of $\mathfrak{g}$-modules

▶ Note that $\sigma(S^k(\mathfrak{g})) \subseteq F_k U(\mathfrak{g})$, i.e. σ is compatible with filtrations. Hence, it gives rise to
$$\mathrm{gr}(\sigma): \underbrace{\mathrm{gr}\, S(\mathfrak{g})}_{\simeq S(\mathfrak{g})} \longrightarrow \mathrm{gr}\, U(\mathfrak{g})$$
which clearly coincides with ϕ, hence, $\mathrm{gr}(\sigma)$ is a vector space isomorphism. Then, by [Homework 4, Problem 4b)], σ is a vector space isom. And by above Exercise, it intertwines $\mathfrak{g}$-actions. ∎

Combining this Lemma and isomorphism $U(\mathfrak{g})^{\mathfrak{g}} \simeq \overbrace{Z U(\mathfrak{g})}^{\text{center of } U(\mathfrak{g})}$, we get:

Corollary 2: $S(\mathfrak{g})^{\mathfrak{g}} \xrightarrow{\sim} Z U(\mathfrak{g})$

Remark 1: Can use this and $\mathfrak{sl}_2 \simeq \mathfrak{so}_3$ to deduce [Hwk 4, Problem 5b)].

Hard Result (Duflo theorem, proved by Kontsevich): For any fin. dim. Lie algebra $\mathfrak{g}$, there is an algebra isomorphism $S(\mathfrak{g})^{\mathfrak{g}} \xrightarrow{\sim} Z U(\mathfrak{g})$

Lecture #9

• Free Lie algebras

Similarly to how finitely generated associative algebras are quotients of the free associative algebras, any f.n. gen-d Lie algebra is a quotient of a free Lie algebra.

Def 1: Let V be a vector space over $\mathbb{k}$. The free Lie algebra $L(V)$ generated by V is the Lie subalgebra of $T(V)$ generated by V.

Placing all el-s of V in degree 1, we see that $L(V)$ is $\mathbb{Z}_{>0}$-graded:

$$\boxed{L(V) = \bigoplus_{n>0} L_n(V), \quad L_n(V) = \text{span of commutators of } n \text{ elements of } V \text{ inside } T(V)}$$

Proposition 1: a) Consider the algebra homom. $\psi: \mathcal{U}(L(V)) \to T(V)$ induced by the embedding $L(V) \hookrightarrow T(V)$. Then ψ is an isomorphism

b) (universal property of $L(V)$) For any Lie algebra $\mathfrak{g}$ over $\mathbb{k}$:

$$\underset{\substack{\uparrow \\ \text{restriction to } V \subseteq L(V)}}{\text{res}}: \operatorname{Hom}_{Lie}(L(V), \mathfrak{g}) \to \operatorname{Hom}_{\mathbb{k}}(V, \mathfrak{g}) \quad \text{is an isomorphism}$$

▶ a) As $L(V)$ is generated by V under the Lie bracket, then $\mathcal{U}(L(V)) \simeq T(V)/I \leftarrow$ some 2-sided ideal. But then $\psi: T(V)/I \xrightarrow[\text{alg. hom}]{} T(V)$ is identity on V, hence, $I = 0$.

b) Any linear map $\alpha: V \to \mathfrak{g}$ can be viewed as $\alpha: V \to \underset{\text{ass. alg}}{\underline{\mathcal{U}(\mathfrak{g})}}$, hence, is the same as an alg. homom $\alpha: T(V) \to \mathcal{U}(\mathfrak{g})$

By part a), $T(V) \simeq \mathcal{U}(L(V))$, and by universal property of $\mathcal{U}(?)$, we get $\alpha: L(V) \to \mathcal{U}(\mathfrak{g})$

But as $L(V)$ is Lie-generated by V and image of V is in $\mathfrak{g} \subset \mathcal{U}(\mathfrak{g})$, we end up with $\alpha: L(V) \to \mathfrak{g}$.

Easy: this assignment is inverse to res $\leftarrow$ Check it! ∎

Exercise: Show that if $n := \dim_{\mathbb{k}}(V)$, then the sequence $d_m(n) := \dim_{\mathbb{k}}(L_m(V))$ (dimensions of the graded pieces of $L(V)$) is uniquely determined by:

$$\boxed{\prod_{m\geq 1} (1-q^m)^{d_m(n)} = 1 - nq}, \quad q = \text{formal variable}$$

Lecture #9

- Baker–Campbell–Hausdorff formula

Let's recall that when we introduced Lie bracket on T_1G, we used only the first nontrivial piece of multiplication map in logarithmic coordinates

$$\boxed{\begin{aligned} &\mu: U\times U \to \mathfrak{g} \\ &(x,y)\mapsto \log(\exp(x)\overset{\text{product in } G}{\cdot}\exp(y)) \\ &\mu(x,y) = x+y+\tfrac{1}{2}[x,y] + \sum_{n\geq 3}\underbrace{\mu_n(x,y)}_{\text{degree } n \text{ terms}} \end{aligned}}$$

← see Lecture 6

A natural question to ask is:

Q: Did we loose any information in $Lie(G)$ arising through $\{\mu_n\}_{n\geq 3}$?

Surprisingly (or rather not), it turns out that all μ_n can be expressed via $[,]$.

Theorem 1: For any n, the expression $\mu_n(x,y)$ is a Lie polynomial in x,y of degree n with $\mathbb{Q}$-coefficients (i.e. $\mathbb{Q}$-linear combination of iterated commutators of x,y), which is universal (i.e. G-independent)

The short proof of this result requires a construction of coproduct on $U(\mathfrak{g})$.
First, define $\Delta: T(\mathfrak{g})\to T(\mathfrak{g})\otimes T(\mathfrak{g})$ as an algebra homomorphism with

$$\boxed{\Delta(x) = x\otimes 1 + 1\otimes x \quad \forall x\in\mathfrak{g}}$$

← $\mathfrak{g}$ - any Lie algebra

Lemma 2: Δ descends to an algebra homom. $\Delta: U(\mathfrak{g})\to U(\mathfrak{g})\otimes U(\mathfrak{g})$

► It suffices to prove $\Delta(I)\subseteq I\otimes T(\mathfrak{g}) + T(\mathfrak{g})\otimes I$ for $I = \langle xy-yx-[x,y] \mid x,y\in\mathfrak{g}\rangle$
But, we have:

$$\begin{aligned}\Delta(xy-yx-[x,y]) &= (x\otimes 1+1\otimes x)(y\otimes 1+1\otimes y) - (y\otimes 1+1\otimes y)(x\otimes 1+1\otimes x) \\ &\quad - ([x,y]\otimes 1 + 1\otimes [x,y]) \\ &= (xy-yx-[x,y])\otimes 1 + 1\otimes (xy-yx-[x,y])\end{aligned}$$

■

These algebra homomorphisms are called *coproducts*

$$\boxed{\Delta: T(\mathfrak{g})\to T(\mathfrak{g})\otimes T(\mathfrak{g}), \quad \Delta: U(\mathfrak{g})\to U(\mathfrak{g})\otimes U(\mathfrak{g})}$$

Def 2: An element x of $T(\mathfrak{g})$ or $U(\mathfrak{g})$ (or their completions w.r.t. filtration) is:

a) *primitive* if $\Delta(x) = x\otimes 1 + 1\otimes x$

b) *group-like* if $\Delta(x) = x\otimes x$