# Lecture #10

- Last time:
  - Some Remarks on $\mathfrak{sl}_2$ theory & PBW thm
  - Symmetrization map
  - Free Lie algebras
  - Baker-Campbell-Hausdorff f-la

- Today we shall start with proving above statement
  - Discuss coproduct $\Delta: T(\mathfrak{g}) \to T(\mathfrak{g}) \otimes T(\mathfrak{g})$, $U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$ & notions of primitive / group-like el-s (see the end of p.5 of Lecture 9 notes)

Proof of Thm (Lect 9)

It suffices to prove the statement for the biggest Lie algebra generated by given two el-s. To this end, let $V = \mathbb{C}^2$ with basis $x, y$, and consider free Lie algebra $L(V)$. By Prop 1 of Lecture 9, $U(L(V)) \simeq \mathbb{C}\langle x,y\rangle$ - free algebra.

Then, all maps $\mu_n$ appearing in BCH f-la are arising through the universal formula:

$$\boxed{\log(\exp(x)\cdot\exp(y)) = \sum_{n\geq 1} \mu_n(x,y) \in \mathbb{C}\langle x,y\rangle^{\wedge}}$$ completion w.r.t. degree

Here, $\exp(X) = \sum_{n\geq 0} \frac{X^n}{n!}$, $\log(X) = -\sum_{n>0} \frac{(1-X)^n}{n}$ and all above is viewed as formal series in noncommuting two variables.

Want: Each $\mu_n(x,y)$, which is some degree $n$ homog. pol-l in $x, y$, is actually a Lie polynomial, i.e. is an element of $L_n(V)$.

The proof follows from the following exercise:

Exercise: a) El-t $u$ - primitive $\Leftrightarrow$ $\exp(u)$ - group-like
b)* Primitive elements of $U(\mathfrak{g})$ are just elements of $\mathfrak{g}$ ($\forall$ Lie algebra $\mathfrak{g}$) assuming char $\mathbb{K} = 0$
c) Provide counterexample to b) when char $\mathbb{K} = p > 0$

$x, y \in V \subseteq L(V) \overset{a)}{\Rightarrow} \exp(x), \exp(y)$ - group-like $\overset{a)}{\Rightarrow} \log(\exp(x)\cdot\exp(y))$ - primitive
$\Rightarrow \mu_n(x,y)$ - primitive $\overset{b)}{\Rightarrow} \mu_n(x,y) \in L_n(V)$.

Exercise: Verify that $\mu_3(x,y) = \frac{1}{12}([x,[x,y]] + [y,[y,x]])$

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- For the rest of today, we shall discuss solvable and nilpotent Lie algebras.

Recall: $I \subseteq \mathfrak{g}$ is an ideal if $[x,y] \in I \quad \forall x \in I, y \in \mathfrak{g}$.

We start with the following two simple results:

**Lemma 1**: For any Lie algebra homomorphism $\varphi: \mathfrak{g}_1 \to \mathfrak{g}_2$:
a) $\operatorname{Ker}\varphi$ is an ideal of $\mathfrak{g}_1$
b) $\operatorname{Im}\varphi$ is a Lie subalgebra of $\mathfrak{g}_2$
c) $\mathfrak{g}_1/\operatorname{Ker}\varphi \xrightarrow{\sim} \operatorname{Im}\varphi$ as Lie alg.

**Lemma 2**: Given two ideals $I_1, I_2$ of $\mathfrak{g}$, the following are also ideals in $\mathfrak{g}$:
a) $I_1 + I_2 := \{x+y \mid x \in I_1, y \in I_2\}$
b) $[I_1, I_2] := \operatorname{span}\{[x,y] \mid x \in I_1, y \in I_2\}$
c) $I_1 \cap I_2$

[ Easy Exercise: Prove both lemmas

**Def 1**: The commutant of a Lie algebra $\mathfrak{g}$ is the ideal $[\mathfrak{g},\mathfrak{g}]$

Obvious: a) $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$ – abelian Lie alg.
b) If $\mathfrak{g}/I$ – abelian, then $I \supseteq [\mathfrak{g},\mathfrak{g}]$

[ Example/Exercise: $[\mathfrak{gl}_n(\mathbb{K}), \mathfrak{gl}_n(\mathbb{K})] = \mathfrak{sl}_n(\mathbb{K}) = [\mathfrak{sl}_n(\mathbb{K}), \mathfrak{sl}_n(\mathbb{K})]$

**Def 2**: For a Lie algebra $\mathfrak{g}$, the sequence of ideals
$$\mathfrak{g} = D^0\mathfrak{g} \supseteq D^1\mathfrak{g} \supseteq D^2\mathfrak{g} \supseteq \dots \quad \text{with } \boxed{D^{i+1}\mathfrak{g} = [D^i\mathfrak{g}, D^i\mathfrak{g}]}$$
is called the derived sequence

**Def 3**: Lie algebra $\mathfrak{g}$ is solvable if $D^n\mathfrak{g} = 0$ for $n \gg 1$.

**Lemma 3**: $\mathfrak{g}$ is solvable
$\Updownarrow$
$\exists$ sequence of subalgebras $\mathfrak{g} = \mathfrak{a}^0 \supset \mathfrak{a}^1 \supset \mathfrak{a}^2 \supset \dots \supset \underset{=0}{\mathfrak{a}^n}$ such that $\forall i$ $\mathfrak{a}^{i+1}$ is an ideal in $\mathfrak{a}^i$ and $\mathfrak{a}^i/\mathfrak{a}^{i+1}$ – abelian

$\Rightarrow$: take $\mathfrak{a}^i = D^i\mathfrak{g}$

$\Leftarrow$: Argue by induction that $D^i\mathfrak{g} \subseteq \mathfrak{a}^i \quad \forall i$

$\blacksquare$

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**Def 4**: For a Lie algebra $\mathfrak{g}$, the sequence of ideals
$$\mathfrak{g} = D_0\mathfrak{g} \supseteq D_1\mathfrak{g} \supseteq D_2\mathfrak{g} \supseteq \dots \quad \text{with } \boxed{D_{i+1}\mathfrak{g} = [\mathfrak{g}, D_i\mathfrak{g}]}$$
is called the lower central series.

**Def 5**: Lie algebra $\mathfrak{g}$ is nilpotent if $D_n\mathfrak{g} = 0$ for $n \gg 1$.

**Lemma 4**: $\mathfrak{g}$ – nilpotent
$$\Updownarrow$$
$\exists$ sequence of ideals $\mathfrak{g} = \mathfrak{a}_0 \supseteq \mathfrak{a}_1 \supseteq \mathfrak{a}_2 \dots \supseteq \mathfrak{a}_n = 0$ such that $[\mathfrak{g}, \mathfrak{a}_i] \subseteq \mathfrak{a}_{i+1}$

$\Rightarrow$: set $\mathfrak{a}_i = D_i\mathfrak{g}$

$\Leftarrow$: Show that $\mathfrak{a}_i \supseteq D_i\mathfrak{g}$ $\forall i$ ■

**Exercise**: a) Any subalg or quotient of a solvable Lie algebra are solvable

b) —"— nilpotent —"— nilpotent

c) If ideal $I$ of $\mathfrak{g}$ and quotient $\mathfrak{g}/I$ are solvable, must $\mathfrak{g}$ be solvable?

d) —"— nilpotent, —"— nilpotent?

**Example/Exercise**: $\mathfrak{b} := \{$upper-triangular $n\times n$ matrices$\}$ – solvable But Not nilpotent

$\mathfrak{n} := \{$strictly upper-triangular $n\times n$ matrices$\}$ – nilpotent

The two basic results on solvable & nilpotent Lie algebras are:

**Theorem 1 (Lie's theorem)**: Let $\rho: \mathfrak{g} \to \mathfrak{gl}(V)$ be a complex fin. dim. representation of a solvable Lie algebra $\mathfrak{g}$. Then, there is a basis of $V$ s.t. all $\rho(x)$ are upper-triangular with respect to this basis

**Theorem 2 (Engel's theorem)**: A Lie algebra $\mathfrak{g}$ is nilpotent if and only if $\forall x \in \mathfrak{g}$ the operation $\mathrm{ad}(x): \mathfrak{g} \to \mathfrak{g}$ is nilpotent

Rmk: 1) Lie's theorem basically says that image of any solvable Lie alg $\mathfrak{g}$ is inside $\mathfrak{b}$.

2) Engel's theorem explains the terminology "nilpotent" is not just an accident

As an immediate Corollary, we get:

**Corollary 1**: a) Any irreducible repr-n of a solvable Lie algebra is 1-dim.
b) If $\mathfrak{g}$ is solvable $\mathbb{C}$ Lie algebra, then $\exists$ ideals $0 \subseteq \mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \ldots \subseteq \mathfrak{a}_N = \mathfrak{g}$ such that $\mathfrak{a}_{k+1}/\mathfrak{a}_k$ is 1-dim
c) $\mathfrak{g}$ - solvable $\Leftrightarrow$ $[\mathfrak{g}, \mathfrak{g}]$ - nilpotent.

▷ a) Obvious

b) Apply Lie theorem to $\mathrm{ad}: \mathfrak{g} \curvearrowright \mathfrak{g}$ (and note that ideals = submodules of $\mathfrak{g}$)

c) $\Leftarrow$: if $[\mathfrak{g},\mathfrak{g}]$ - nilpotent, then it's also solvable
$\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$ - abelian, hence, solvable $\Big\} \Rightarrow \mathfrak{g}$ - solvable

$\Rightarrow$: Consider $\mathrm{ad}: \mathfrak{g} \curvearrowright \mathfrak{g}$. By Lie theorem, find a basis s.t. $\mathrm{ad}(\mathfrak{g}) \subseteq \mathfrak{b}$ ← upper-△

Then $\mathrm{ad}([\mathfrak{g},\mathfrak{g}]) \subseteq [\mathfrak{b},\mathfrak{b}] \underset{\uparrow \text{Exercise}}{=} \mathfrak{n}$ - nilpotent $\Rightarrow$ $\mathrm{ad}([\mathfrak{g},\mathfrak{g}])$ - nilpotent.

As central el-s do not affect the lower central series $\Rightarrow [\mathfrak{g},\mathfrak{g}]$ - nilpotent ∎

• The proofs of both theorems are based on the following two results:

**Proposition 1**: Let $\rho: \mathfrak{g} \to \mathfrak{gl}(V)$ be a complex f. dim. module of a solvable $\mathfrak{g}$.
Then $\exists v \in V \setminus \{0\}$ - common eigenvector of all $\rho(x)$ $\forall x \in \mathfrak{g}$

Arguing by induction in $\dim(V)$ and passing to the quotient module $V/\mathbb{C}v$, we get
Prop 1 $\Rightarrow$ Thm 1

**Proposition 2**: Let $V$ be a f. dim. vector space, and $\underset{\nwarrow \text{any Lie algebra}}{\mathfrak{g} \subseteq \mathfrak{gl}(V)}$ consist of nilpotent operators. Then $\exists v \in V \setminus \{0\}$ which is annihilated by all elements of $\mathfrak{g}$.

Arguing by induction in $\dim(V)$, we find a basis of $V$ s.t. all elements of $\mathfrak{g}$ are strictly upper-triangular. Applying this to $\mathrm{ad}: \mathfrak{g} \curvearrowright \mathfrak{g}$, we see

Prop 2 $\Rightarrow$ Thm 2.

So: It remains to prove the above Prop 1 & Prop 2.

We shall do Prop 1 today and Prop 2 next time.

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Proof of Proposition 1

We shall prove by induction on $\dim \mathfrak{g}$. Base case ($\dim \mathfrak{g} = 1$) is clear.

- Consider $\mathfrak{g} \supseteq [\mathfrak{g}, \mathfrak{g}]$. As $\mathfrak{g}$ - solvable $\Rightarrow \mathfrak{g} \neq [\mathfrak{g},\mathfrak{g}]$

  Pick any $\mathfrak{g}'$ - subspace of codim 1 in $\mathfrak{g}$ containing $[\mathfrak{g},\mathfrak{g}]$: $[\mathfrak{g},\mathfrak{g}] \subseteq \mathfrak{g}' \overset{\text{codim } 1}{\subsetneq} \mathfrak{g}$

  As $[\mathfrak{g}, \mathfrak{g}'] \subseteq [\mathfrak{g},\mathfrak{g}] \subseteq \mathfrak{g}'$, $\mathfrak{g}'$ is an ideal. Let $x \in \mathfrak{g}$ be such that $\mathfrak{g} \overset{\text{v.sp.}}{=} \mathfrak{g}' \oplus \mathbb{C}x$

  Induction hypothesis applied to $\mathfrak{g}' \curvearrowright V$ implies that

  $$\boxed{\exists v \in V \setminus \{0\} \text{ s.t. } \rho(h) = \lambda(h) \cdot v \quad \forall h \in \mathfrak{g}' \quad \text{with } \lambda \in (\mathfrak{g}')^*}$$

- Consider $\boxed{W := \operatorname{span}\{v_0 = v,\ v_1 := \rho(x) v,\ v_2 := \rho(x) v_1, \ldots, v_n = \rho(x) v_{n-1} = \rho(x)^n v, \ldots\}}$

  Then: $\forall h \in \mathfrak{g}'$, have

  $$(*) \quad \boxed{\rho(h) v_k = \lambda(h) v_k + \sum_{\ell < k} \# \cdot v_\ell}, \quad \# \text{ - some coeff.}$$

  Proof is by induction on $k$. Base $k=0$ is clear.

  For $k > 0$, have:

  $$\rho(h) v_k = \rho(h)\rho(x) v_{k-1} = \rho(x) \underbrace{\rho(h) v_{k-1}}_{= \lambda(h) v_{k-1} + \sum_{\ell < k-1} \# \cdot v_\ell} + \overbrace{\rho(\underbrace{[h,x]}_{\in \mathfrak{g}'}) v_{k-1}}^{\lambda([h,x]) \cdot v_{k-1} + \sum_{\ell < k-1} \# \cdot v_\ell} = \lambda(h) v_k + \sum_{\ell < k} \# \cdot v_\ell$$

  ∎

  As $\dim W \le \dim V < \infty$, pick $n$ s.t. $v_0, v_1, \ldots, v_n$ - lin. indep, but $v_{n+1} \in \operatorname{Span}\{v_0, \ldots, v_n\}$

  Then: $\boxed{v_0, v_1, \ldots, v_n \text{ - basis of } W.}$

- Note that it follows from above: $\boxed{\operatorname{tr}_W \rho(h) = (n+1)\, \lambda(h) \quad \forall h \in \mathfrak{g}'}$

  In particular, $\forall h \in \mathfrak{g}'$ as $[h, x] \in \mathfrak{g}'$ and $\operatorname{tr}[A,B] = 0$, we get

  $$(n+1)\lambda([h,x]) = \operatorname{tr}_W(\rho[h,x]) = \operatorname{tr}_W([\rho(h), \rho(x)]) = 0 \Rightarrow \boxed{\lambda([h,x]) = 0 \ \forall h \in \mathfrak{g}'}$$

- But then: getting back to the proof of $(*)$, we actually get

  $$\boxed{\rho(h) v_k = \lambda(h) v_k \quad \forall k \ \forall h \in \mathfrak{g}'}$$

- Picking $w \in W \setminus \{0\}$ to be an eigenvector of $\rho(x)$, we see that it's also an eigenvector of all $\{\rho(h) \mid h \in \mathfrak{g}'\}$. Hence: $w$ - common eigenvector of all $\{\rho(x) \mid x \in \mathfrak{g}\}$

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