Lecture #11

- Last time
  - coproduct on $U(\mathfrak{g})$ and a quick proof of BCH formula
  - nilpotent and solvable Lie algebras
  - Lie theorem and Engel theorem
    - ↳ Corollary 1 from Lecture 10
    - → Proofs are based on two simpler results (Prop 1, Prop 2 from Lecture 10).

- We shall start today's class with the above Propositions

> **Prop 1**: Let $\rho: \mathfrak{g} \to \mathfrak{gl}(V)$ be a complex fin. dim-l module of a solvable Lie algebra $\mathfrak{g}$.
> Then $\exists\, v \in V \setminus \{0\}$, s.t. $\rho(x)v = \lambda(x)v \quad \forall x \in \mathfrak{g}$, with $\lambda \in \mathfrak{g}^*$

- Finish the last step of the proof we presented in the end of last class
- Discuss Corollary 1, and recall how Prop 1 $\Rightarrow$ Lie thm.

> **Prop 2**: Let V be a fin. dim. complex vector space, and $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ be a Lie algebra (i.e. closed under $[-,-]$) consisting of nilpotent operators. Then
> $\exists\, v \in V \setminus \{0\}$ s.t. $\rho(x)v = 0 \quad \forall x \in \mathfrak{g}$.

Rmk: By extending $\underset{\mathbb{R}}{\otimes} \mathbb{C}$, the above result clearly holds over $\mathbb{R}$ as well.

Proof

- We shall prove by induction on $\dim(\mathfrak{g})$ again. Base case ($\dim \mathfrak{g} = 1$) is clear.

Key Step in the proof is:

$\boxed{\text{Claim: There is an ideal } \mathfrak{h} \subseteq \mathfrak{g} \text{ of codimension } 1}$

- First, let's finish the proof using this Claim. By the induction hypothesis (for $\mathfrak{h}$):
$V^{\mathfrak{h}} = \{v \in V \mid \rho(x)v = 0 \ \forall x \in \mathfrak{h}\} \neq 0$ (see Hwk 4, Problem 1c for notation). As in the proof, pick $x \in \mathfrak{g}$ so that $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{C}x$. Since $\mathfrak{h}$ - ideal, have $[x, \mathfrak{h}] \subseteq \mathfrak{h} \Rightarrow ad(x)(V^{\mathfrak{h}}) \subseteq V^{\mathfrak{h}}$

- As in the proof of Prop 1, take any $v \in V^{\mathfrak{h}} \setminus \{0\}$ and consider the sequence
$v_0 = v,\ v_1 = \rho(x)v,\ v_2 = \rho(x)v_1 = \rho(x)^2 v, \longrightarrow v_n = \rho(x)^n v, \ldots$
As $\rho(x)$ - nilpotent $\exists\, N > 0$ s.t. $v_{N-1} \in V^{\mathfrak{h}} \setminus \{0\}$ and $v_N = 0$.

THUS: v_{N-1} is a nonzero vector of V annihilated both by $ad(x)$ and $ad(\mathfrak{h})$ $\Rightarrow \boxed{v_{N-1} \in V^{\mathfrak{g}}}$

It remains to prove the Claim above.

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(Continuation – proof of the Claim)

Consider the maximal proper Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, i.e. $\mathfrak{h} \subsetneq \mathfrak{g}$ is closed under $[\,,\,]$ and there is no $\mathfrak{h}'$ s.t. $\mathfrak{h} \subsetneq \mathfrak{h}' \subsetneq \mathfrak{g}$ with $\mathfrak{h}'$ closed under $[-,-]$. Clearly such $\mathfrak{h}$ does exist as $\mathfrak{g}$ is fin. dim and $\forall x \in \mathfrak{g}$ $\mathbb{C}x$ is a subalgebra.

As we shall now see, $\mathfrak{h}$ is actually an ideal of $\mathfrak{g}$ and $\dim(\mathfrak{g}/\mathfrak{h}) = 1$.

To this end, $\forall y \in \mathfrak{h}$ the operator $\underset{\uparrow \text{ nilpotent}}{\mathrm{ad}(y)} : \mathfrak{g} \to \mathfrak{g}$ descends to $\underset{\uparrow \text{ nilpotent}}{\mathrm{ad}(y)} : \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}/\mathfrak{h}$

By the induction in $\dim(V)$, there is $\underset{\neq 0}{\bar{x}} \in \mathfrak{g}/\mathfrak{h}$ s.t. $[y, \bar{x}] = 0 \in \mathfrak{g}/\mathfrak{h}$

Picking any lift $x \in \mathfrak{g}$ of $\bar{x} \in \mathfrak{g}/\mathfrak{h}$, we see $[y, x] \in \mathfrak{h}$ $\forall y \in \mathfrak{h}$. Hence:
span $\{\mathfrak{h}, x\}$ is a Lie subalgebra of $\mathfrak{g}$.

Thus, by maximality of $\mathfrak{h}$, we see that $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{C}x$ (i.e. $\mathfrak{h}$ is of codim 1 in $\mathfrak{g}$) as well as $\mathfrak{h} \subseteq \mathfrak{g}$ – ideal ∎

• For the rest of today, we shall discuss semisimple and reductive algebras.

|| Def 1: A Lie algebra $\mathfrak{g}$ is semisimple if it doesn't contain nonzero solvable ideals.

|| Def 2: A Lie algebra $\mathfrak{g}$ is simple if it is not abelian and has no nonzero proper ideals

Rmks: a) the condition "not abelian" in Def 2 is to exclude 1-dim Lie algebras

b) if $\mathfrak{g}$ – semisimple, then $\mathfrak{z}(\mathfrak{g}) = 0$

c) $\mathfrak{g}$ – simple $\Rightarrow$ $\mathfrak{g}$ – semisimple

▹ If not, then $\mathfrak{g}$ – solvable $\Rightarrow [\mathfrak{g},\mathfrak{g}] \subseteq \mathfrak{g}$ – proper ideal $\Rightarrow [\mathfrak{g},\mathfrak{g}] = 0 \Rightarrow \mathfrak{g}$ – abelian $\Rightarrow$ ↯ ∎

First Example: $\mathfrak{g} = \mathfrak{sl}_2$ – simple

▹ Discuss why. Use $\mathrm{ad}(h) : \mathfrak{sl}_2 \to \mathfrak{sl}_2$ with pairwise distinct eigenvalues ∎

[We shall soon see that all $\mathfrak{sl}_n, \mathfrak{so}_n, \mathfrak{sp}_{2n}$ are in fact simple.

> Lemma 1: In any fin. dimensional Lie algebra $\mathfrak{g}$, there is a unique solvable ideal which contains all other solvable ideals.

▹ Uniqueness is clear. Existence follows by just taking sum of all solvable ideals. It suffices to check that if I_1, I_2 are solvable ideals, then so is $I_1 + I_2$.

But: $I_1 \underset{\text{ideal}}{\subseteq} I_1 + I_2$ and $\underbrace{I_2 \twoheadrightarrow}_{\text{solvable} \Rightarrow} \underbrace{(I_1+I_2)/I_1}_{\text{solvable}} \underset{\text{Hwk 5 Problem 4c}}{\Longrightarrow} I_1 + I_2$ – solvable ∎

|| Def 3: The maximal solvable ideal of $\mathfrak{g}$ (from Lemma 1) is called the radical: $\mathrm{rad}(\mathfrak{g})$

(so $\mathfrak{g}$ – semisimple $\Leftrightarrow \mathrm{rad}(\mathfrak{g}) = 0$)

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Exercise: a) Prove $\mathrm{rad}(\mathfrak{g} \oplus \mathfrak{h}) = \mathrm{rad}(\mathfrak{g}) \oplus \mathrm{rad}(\mathfrak{h})$.

b) Deduce that direct sum of semisimple Lie alg-s is semisimple.

Lemma 2: a) For any $\mathfrak{g}$, the quotient $\mathfrak{g}_{ss} := \mathfrak{g}/\mathrm{rad}(\mathfrak{g})$ is semisimple, so that

$$(*) \quad 0 \to \underbrace{\mathrm{rad}(\mathfrak{g})}_{\text{solvable}} \to \mathfrak{g} \to \underbrace{\mathfrak{g}_{s.s}}_{\text{semisimpl}} \to 0 \qquad \leftarrow \text{short exact sequence (s.e.s)}$$

b) Vice versa, if $\mathfrak{a} \subseteq \mathfrak{g}$ is a solvable ideal s.t. $\mathfrak{g}/\mathfrak{a}$ – semisimple $\Rightarrow \mathfrak{a} = \mathrm{rad}(\mathfrak{g})$

a) For any solvable ideal $\bar{I} \subseteq \mathfrak{g}/\mathrm{rad}(\mathfrak{g})$, consider its preimage $I = \pi^{-1}(\bar{I})$, where $\pi : \mathfrak{g} \twoheadrightarrow \mathfrak{g}/\mathrm{rad}(\mathfrak{g})$. Then, we have a s.e.s. $0 \to \underbrace{\mathrm{rad}(\mathfrak{g})}_{\text{solvable}} \to I \to \underbrace{\bar{I}}_{\text{solvable}} \to 0 \Rightarrow$
$\Rightarrow I$ – solvable $\Rightarrow I = \mathrm{rad}(\mathfrak{g})$ and $\bar{I} = 0$.

b) Know $\mathfrak{a} \subseteq \mathrm{rad}(\mathfrak{g})$. If $\mathfrak{a} \neq \mathrm{rad}(\mathfrak{g})$, then the image $\mathrm{rad}(\mathfrak{g}) \subseteq \mathfrak{g}/\mathfrak{a}$ – solvable ideal $\Rightarrow$ ↯ ∎

In fact, the above result can be significantly upgraded! (proof is omitted for now)

Theorem (Levi theorem): Any fin. dim. Lie algebra $\mathfrak{g}$ (over $\mathbb{k}$ of characteristic zero) can be written as a direct sum $\mathfrak{g} \overset{\text{v.sp.}}{=} \underbrace{\mathrm{rad}(\mathfrak{g})}_{\text{ideal}} \oplus \underbrace{\mathfrak{g}_{s.s.}}_{\text{subalgebra NOT ideal}}$ with $\mathfrak{g}_{ss}$ being semisimple

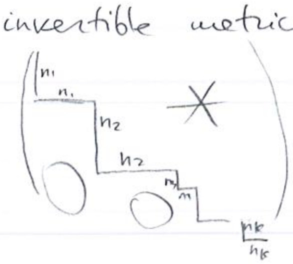
Such decomposition of $\mathfrak{g}$ is called the *Levi decomposition*

Remarks: a) the above thm means that $\mathfrak{g} \twoheadrightarrow \mathfrak{g}_{s.s}$ in $(*)$ admits a (non-unique) splitting $\mathfrak{g}_{s.s.} \hookrightarrow \mathfrak{g}$

b) In view of [Hwk 5, Problem 7], we can recast above by saying

$$\boxed{\mathfrak{g} \simeq \mathfrak{g}_{ss} \ltimes \mathrm{rad}(\mathfrak{g})} \leftarrow \text{semidirect product of Lie algebras.}$$

Exercise: Following [Hwk 2, Problem 6(a)] consider a parabolic subgp $P_{n_1,\dots,n_k} \subseteq GL(n_1+\dots+n_k)$ consisting of invertible matrices of the form



and let $\mathfrak{g} = \mathrm{Lie}(P_{n_1,\dots,n_k})$

a) Compute $\mathrm{rad}(\mathfrak{g})$

b) Provide an example of the Levi decomposition

Proposition 1: Let V be an irreducible complex $\mathfrak{g}$-module. Then:
a) $\forall h \in \mathrm{rad}(\mathfrak{g})$, $\rho(h)$ is a scalar operator on V
b) $\forall h \in [\mathfrak{g}, \mathrm{rad}(\mathfrak{g})]$, $\rho(h)$ acts by zero.

▷ a) Consider $\underbrace{\mathrm{rad}(\mathfrak{g})}_{\text{solvable}} \curvearrowright V \Rightarrow$ By Lie theorem $\exists \overset{\neq 0}{v} \in V$ s.t.
$$\boxed{\rho(h)v = \lambda(h)\cdot v \quad \forall h \in \mathrm{rad}(\mathfrak{g}) \text{ and some } \lambda \in \mathrm{rad}(\mathfrak{g})^*}$$

Consider the subspace
$$W_\lambda := \{w \in V \mid \rho(h)w = \lambda(h)\cdot w \quad \forall h \in \mathrm{rad}(\mathfrak{g})\}$$
It's nonzero, as $v \in W_\lambda$.
Furthermore, similarly to our proof of Lie theorem we get:
$$\rho(x)(W_\lambda) \subseteq W_\lambda \quad \forall x \in \mathfrak{g}$$
(Exercise: prove this)
So: $\overset{\neq 0}{W_\lambda}$ – submodule of $V \overset{V\text{-irred}}{\Longrightarrow} W_\lambda = V \Rightarrow$ a) follows.

b) If $x \in [\mathfrak{g}, \mathrm{rad}(\mathfrak{g})] \subseteq \mathrm{rad}(\mathfrak{g}) \Rightarrow \rho(x) = c\cdot \mathrm{Id} \Rightarrow \underset{\overset{\|}{0}}{\mathrm{tr}_V \rho(x)} = c\cdot \dim V$ as $\mathrm{tr}([A,B]) = 0$ $\Rightarrow c = 0$ ■

Thus, on the level of irred. repr-s the radical acts in a simple way while $[\mathfrak{g}, \mathrm{rad}(\mathfrak{g})]$ acts by zero.

|| **Def 4**: A Lie algebra $\mathfrak{g}$ is *reductive* if $\mathrm{rad}(\mathfrak{g}) = \mathfrak{z}(\mathfrak{g})$ ← center of $\mathfrak{g}$ $= \{x \in \mathfrak{g} \mid [x,y] = 0 \ \forall y \in \mathfrak{g}\}$

So $\mathfrak{g}$ is reductive if $\mathfrak{g}/\mathfrak{z}(\mathfrak{g})$ – semisimple.

Example: If $\mathfrak{a}$ – abelian, $\mathfrak{b}$ – s.s. Lie alg $\Rightarrow \mathfrak{a} \oplus \mathfrak{b}$ is reductive, as $(\mathfrak{a} \oplus \mathfrak{b})/\mathfrak{a} \simeq \mathfrak{b}$ – s.s.

But Levi theorem implies the converse (a different proof will be given later):

Proposition 2: $\mathfrak{g}$ – reductive $\Longleftrightarrow \mathfrak{g} \simeq \underbrace{\begin{pmatrix}\text{abelian} \\ \text{Lie alg}\end{pmatrix}}_{\mathfrak{z}(\mathfrak{g})} \oplus \underbrace{\begin{pmatrix}\text{semisimple} \\ \text{Lie alg}\end{pmatrix}}_{\mathfrak{g}_{ss}}$

Rmk: It's a direct sum, not just $\rtimes$