Lecture # 12

- Last time:
  - finished proof of Lie theorem
  - proved Engel's theorem
  - introduced notions of semisimple, simple, reductive Lie alg-s
  - radical $\mathrm{rad}(\mathfrak{g})$ and Levi theorem
  - finished with decomposition of reductive Lie alg-s (as direct sum!)

$$\mathfrak{g} \simeq \underbrace{\mathfrak{z}(\mathfrak{g})}_{\text{abelian}} \oplus \underbrace{\mathfrak{g}_{s.s}}_{\text{semisimple}}$$

(we shall get a different proof of this later)

Example: $\mathfrak{g} = \mathfrak{gl}_n \rightsquigarrow \mathfrak{g} = \mathbb{C}\cdot \mathrm{Id} \oplus \mathfrak{sl}_n$

- Semisimplicity through invariant bilinear forms

Last time we established simplicity of $\mathfrak{sl}_2$ by brute force (using $\mathrm{ad}(h) : \mathfrak{sl}_2 \circlearrowleft$)
We shall now see how one could check semisimplicity in general.

Exercise: a) Verify that given any $\mathfrak{g}$-module V, the following defines a $\mathfrak{g}$-action on the space of all bilinear forms $V\times V \to \mathbb{k}$:

$$\boxed{xB(v,w) = -B(x.v, w) - B(v, x.w) \quad \forall x\in\mathfrak{g}}$$

We call B to be $\mathfrak{g}$-invariant if $B(x.v,w) + B(v, x.w) = 0 \ \forall x\in\mathfrak{g} \ \forall v,w$

b) Verify that B is $\mathfrak{g}$-invariant if the corresponding linear map

$$V \longrightarrow V^*, \quad v \longmapsto B(v,-)$$

is a $\mathfrak{g}$-module homom. (Check that identifying bil. maps $V\times V \to \mathbb{k}$ with linear maps $V\to V^*$, action in a) agrees with that on $\mathrm{Hom}(V,V^*)$)

c) If V is an irreducible $\mathfrak{g}$-module$/_{\mathbb{C}}$, show that the space of $\mathfrak{g}$-inv. bilinear forms $V\times V \to \mathbb{C}$ is zero or 1-dimensional

d) If $V = \mathfrak{g}$ w.r.t. adjoint action of $\mathfrak{g}$, and $I \subseteq \mathfrak{g}$- ideal, then its orthogonal complement $I^\perp = \{x\in\mathfrak{g} \mid B(x,y)=0 \ \forall y\in I\}$ is also ideal.

Example: Let $\mathfrak{g} = \mathfrak{gl}_n$ and $B(x,y) := \mathrm{tr}(xy)$. It is clearly symmetric and invariant:

$$B([x,y]z + y[x,z]) = \mathrm{tr}(xyz - yxz + yxz - yzx) = \mathrm{tr}([x,yz]) = 0$$

Generalizing this example with the same proof, we have:

Lemma 1. For any $\mathfrak{g}$-module V, the following is a symmetric inv. bilinear form on $\mathfrak{g}$:

$$B_V(x,y) = \mathrm{tr}_V\big(\rho_V(x)\rho_V(y)\big)$$

Lecture #12

The reason why the forms from Lemma 1 are important is the following:

Lemma 2: Let $\mathfrak{g}$ be a Lie alg, V - f. dim. $\mathfrak{g}$-module, s.t. $B_V : \mathfrak{g}\times\mathfrak{g}\to \mathbb{K}$ is nondegenerate. Then: $\mathfrak{g}$ - reductive

Exercise: Given any filtration $0=F_0V\subseteq F_1V\subseteq \ldots \subseteq F_NV=V$ of a $\mathfrak{g}$-module V by submodules, we have $B_V(x,y)=\sum_{k=1}^{N} B_{F_kV/F_{k-1}V}(x,y)$ $\forall x,y\in\mathfrak{g}$

Proof of Lemma 2

▷ To prove $\mathfrak{g}$ - reductive, it suffices to show that $[\mathfrak{g}, \mathrm{rad}(\mathfrak{g})]=0$. Pick any $x\in[\mathfrak{g},\mathrm{rad}(\mathfrak{g})]$. Then by [Lecture 11, Prop 1], $\rho_W(x)=0$ for any irreducible $\mathfrak{g}$-module (W,ρ_W). Thus, $B_W(x,y)=0$ $\forall y\in\mathfrak{g}$ $\forall$ irred. W.

But any fin. dim $\mathfrak{g}$-module V admits a (Jordan - Hölder) filtration $0=F_0V\subseteq F_1V\subseteq \ldots \subseteq F_NV=V$ s.t. each $F_kV/F_{k-1}V$ is an irreducible $\mathfrak{g}$-mod. Then $B_V(x,-)\equiv 0$ by the above Exercise $\Rightarrow$ contradiction! ◼

Now we are ready to state our first key result for today:

Theorem 1: All classical Lie algebras are reductive.

▷
a) For $\mathfrak{g}=\mathfrak{gl}_n(\mathbb{K})$, consider the natural $\mathfrak{g}$-action $\mathfrak{gl}_n(\mathbb{K})\curvearrowright \mathbb{K}^n=:V$
Then $B_V(x,y)=\sum_{i,j=1}^{n} x_{ij}y_{ji}$ is clearly non-degenerate (with $\{E_{ji}\}$ being dual to E_{ij})

b) For $\mathfrak{g}=\mathfrak{sl}_n(\mathbb{K})$, we take the same $V=\mathbb{K}^n$. Then
$B_V(x,y)=0$ for $x\in\mathfrak{sl}_n$, $y\in\mathbb{C}\cdot\mathrm{Id}$, hence, non-degeneracy follows from a)

c) For $\mathfrak{g}=\mathfrak{so}_n(\mathbb{K})$, take $V=\mathbb{K}^n$. Then:
$$B_V(x,y)=\sum_{i,j} x_{ij}y_{ji} \underset{y_{ij}=-y_{ji}}{\overset{x_{ij}=-x_{ji}}{=}} -2\sum_{i>j} x_{ij}y_{ij}$$, which is clearly nondegenerate ◼

Exercise: a) Finish above proof by treating $\mathfrak{g}=\mathfrak{sp}_n(\mathbb{K})$, $\mathfrak{g}=\mathfrak{u}_n$, $\mathfrak{g}=\mathfrak{su}_n$ (explain why other cases follow as well)

b) Verify that $\mathfrak{sl}_n(\mathbb{K})$, $\mathfrak{so}_n(\mathbb{K})$ with $n>2$, $\mathfrak{su}_n$, $\mathfrak{sp}_{2n}$ are semisimple, while $\mathfrak{gl}_n(\mathbb{K})=\mathbb{K}\cdot\mathrm{Id}\oplus\mathfrak{sl}_n(\mathbb{K})$, $\mathfrak{u}_n=i\mathbb{R}\cdot\mathrm{Id}\oplus\mathfrak{su}_n$

Lecture #12

As an important example of the above setup, we can consider $\mathfrak{g} \overset{ad}{\curvearrowright} \mathfrak{g} =: V$.

Def 1: The Killing form on $\mathfrak{g}$ is the bilinear form $B_{\mathfrak{g}}$ from above, i.e.

$$\boxed{K(x,y) = tr_{\mathfrak{g}}(ad\,x \circ ad\,y)}$$

Warning: If $\mathfrak{a} \leq \mathfrak{g}$ is a Lie subalgebra, then the Killing form on $\mathfrak{a}$ is NOT the restriction of the on $\mathfrak{g}$ to $\mathfrak{a} \times \mathfrak{a}$. Hence, if needed, we shall use $K^{\mathfrak{a}}$ for the former and $K^{\mathfrak{g}}$ to the latter.

(Exercise: If $\mathfrak{a}$ is an ideal of $\mathfrak{g}$, then $K^{\mathfrak{a}} = K^{\mathfrak{g}}|_{\mathfrak{a}\times\mathfrak{a}}$.

Example: Let's work out example of $\mathfrak{sl}_2$ first of all. Pick a basis e, h, f:

$$ad(e) = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad ad(f) = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}, \quad ad(h) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Thus: $K(e,f) = K(f,e) = 4$, $K(h,e) = K(e,h) = 0$, $K(h,f) = K(f,h) = 0$

$K(h,h) = 8$, $K(e,e) = 0$, $K(f,f) = 0$.

So: $\boxed{K(x,y) = 4\,tr(xy)}$

$\uparrow$ by Exercise 1(c) on p.1, this is not surprising as $\mathfrak{sl}_2$ – simple.

Our other two Key Results for today are Cartan's Criteria for solvability and semisimplicity of Lie algebras

Theorem 2 (Cartan's Criteria for solvability): $\mathfrak{g}$ – solvable iff $K([\mathfrak{g},\mathfrak{g}], \mathfrak{g}) = 0$

Theorem 3 (Cartan's Criteria for semisimplicity):
$\mathfrak{g}$ – semisimple $\Leftrightarrow$ Killing form K is non-degenerate

It is instructive to compare Thm 3 to Lemma 2.

Lecture #12

Let us first deduce Theorem 3 from Theorem 2

Proof of Theorem 3

$\Leftarrow$: K - nondeg, hence, $\mathfrak{g}$ - reductive by Lemma 2. Hence, remains to prove $\mathfrak{z}(\mathfrak{g})=0$.
Pick $x\in \mathfrak{z}(\mathfrak{g})$, then $ad(x)\equiv 0 \Rightarrow K(x,-)\equiv 0 \Rightarrow x\in \mathrm{Ker}\, K \Rightarrow$ ↯

$\Rightarrow$: Let $\mathfrak{g}$ - semisimple, and set $I:=\mathrm{Ker}\, K$. By Exercise 1d), with $I=\mathfrak{g}$, get $I\subseteq \mathfrak{g}$ - ideal $\Rightarrow$ Killing form of I is the restriction of the one on $\mathfrak{g}$.
Thus: $K^I\equiv 0 \xRightarrow{\text{Thm 2}} I$ - solvable, hence, $I=0$ as $\mathfrak{g}$ - semisimple ∎

The proof of Theorem 2 is based on the following general result in linear alg:

Theorem 4 (Jordan decomposition): Let V be a fin. dim. complex vector space

a) Any linear operator A can be uniquely written as

$$A = A_s + A_n, \text{ with } \begin{array}{l} A_n - \text{nilpotent} \\ A_s - \text{semisimple (a.k.a. diagonalizable)} \\ A_s\cdot A_n = A_n\cdot A_s \end{array}$$

b) Define $ad(A): \mathrm{End}(V)\circlearrowleft$ via $ad(A)B := AB-BA = [A,B]$. Then:

$$ad(A)_s = ad(A_s),\quad ad(A)_n = ad(A_n)$$

and $ad(A_s)$ can be written as

$$ad(A_s) = p(ad(A)) \quad \text{for some } p\in t\mathbb{C}[t]$$

c) Define $\bar{A}_s$ that has the same eigenspaces as A_s, but conjugate eigenvalues

$$A_s v = \lambda v \Rightarrow \bar{A}_s v := \bar{\lambda} v.$$

Then $ad(\bar{A}_s)$ can also be written as

$$ad(\bar{A}_s) = q(ad(A)) \text{ for some } q\in t\mathbb{C}[t]$$

We shall now first deduce Thm 2 from Thm 4, and then prove Thm 4

Terminology: The operator $A: V\to V$ is called semisimple if $\forall$ subspace $V'\subseteq V$ s.t. $A(V')\subseteq V'$ $\exists\, V''$ s.t. $A(V'')\subseteq V''$ and $V=V'\oplus V''$.

Lecture #12

Proof of Theorem 2

$\Rightarrow$: If $\mathfrak{g}$-solvable, then by Lie thm there is a basis in which all $ad(x)$ are upper-Δ. But then, any $ad(y)$ with $y \in [\mathfrak{g},\mathfrak{g}]$ is strictly upper-Δ. Therefore, $K(x,y) = tr(ad\,x \circ ad\,y) = 0 \quad \forall x \in \mathfrak{g},\ y \in [\mathfrak{g},\mathfrak{g}]$

$\Leftarrow$: Assume $K([\mathfrak{g},\mathfrak{g}],\mathfrak{g}) = 0$. Let $\mathfrak{a} := ad(\mathfrak{g}) \subseteq End(\mathfrak{g})$. Then, we have $tr(xy) = 0 \quad \forall x \in [\mathfrak{a},\mathfrak{a}], y \in \mathfrak{a}$. As $\mathfrak{g}$ fits into the short exact sequence

$$0 \to \underbrace{\mathfrak{z}(\mathfrak{g})}_{\text{clearly solvable}} \to \mathfrak{g} \to \mathfrak{a} \to 0$$

it suffices to prove $\mathfrak{a}$ is solvable (thus, reducing to subalg. of $gl(?)$). This follows from the following general result:

> **Claim**: Let V be a fin. dim. $\mathbb{C}$ v. space, $\mathfrak{a} \subseteq gl(V)$ - Lie subalgebra s.t. $tr(xy) = 0 \quad \forall x \in [\mathfrak{a},\mathfrak{a}], y \in \mathfrak{a}$. Then: $\mathfrak{a}$ - solvable.

Pick any $x \in [\mathfrak{a},\mathfrak{a}]$ and consider its Jordan decomposition

$$x = x_s + x_n.$$

We also consider $\bar{x}_s$ as in Thm 4c). Then:

$$tr(x \cdot \bar{x}_s) \underset{\substack{\uparrow \\ \text{follows} \\ \text{from proof of Thm 4}}}{=} tr(x_s \cdot \bar{x}_s) = \sum |\lambda_i|^2,$$ where λ_i are eigenvalues of x_s (i.e. gen. eigenv. of x).

But as $x \in [\mathfrak{a},\mathfrak{a}]$, we can write it as $x = \sum [y_j, z_j]$, so that

$$tr(x \cdot \bar{x}_s) = tr\Big(\sum_j [y_j,z_j]\bar{x}_s\Big) = \sum_j tr(y_j[z_j,\bar{x}_s]) = -\sum_j tr(y_j[\bar{x}_s,z_j])$$

However, $[\bar{x}_s, -] = ad(\bar{x}_s) = q(ad\,x)$ by Thm 4c), with $q \in t\mathbb{C}[t]$.

<u>Hence</u>, $[\bar{x}_s, z_j] \in [\mathfrak{a},\mathfrak{a}]$ and so

$$tr(y_j[\bar{x}_s,z_j]) = 0 \Rightarrow tr(x \cdot \bar{x}_s).$$

By above, get $\sum |\lambda_i|^2 = 0 \Rightarrow$ all $\lambda_i = 0 \Rightarrow x_s = 0 \Rightarrow x = x_n$ - nilpotent.

Then, $[\mathfrak{a},\mathfrak{a}]$ - nilpotent Lie algebra (Engel's thm) $\Rightarrow \mathfrak{a}$ - solvable. $\square$

$\blacksquare$

<u>Rmk</u>: While the above proof was over $\mathbb{C}$, the result also holds over $\mathbb{R}$, since both properties are preserved under $\otimes_{\mathbb{R}} \mathbb{C}$.

Lecture #12

Exercise: Let V be a fin. dim. complex vector space.

a) $A: V \to V$ is semisimple $\iff$ A - diagonalizable

b) If $A: V \to V$ is semisimple and $V' \subseteq V$ satisfies $A(V') \subseteq V'$, then then the corresponding operators $V' \to V'$ and $V/V' \to V/V'$ are s.s.

c) If $A, B: V \to V$ are s.s. and $AB = BA$, then $A+B$ is also s.s

d) If $A, B: V \to V$ are nilpotent and $AB = BA$, then $A+B$ is nilpotent

Let us finally prove Theorem 4 (Jordan decomposition).

- Know $V = \bigoplus_{\lambda \in \mathbb{C}} V_\lambda$, V_λ = generalized eigenspace of A with eigenvalue λ, i.e. $(A - \lambda \cdot Id)_{|V_\lambda}$ is nilpotent

Then, we set $A_s : V \to V$ via $A_{s|V_\lambda} = \lambda \cdot Id$, and $A_n := A - A_s$.

Clearly: A_s - s.s. (see Exercise above), A_n - nilpotent, $[A_s, A_n] = 0$.

- If $A = A'_s + A'_n$ is any other such decomposition, then A'_s, A'_n commute with A and hence with A_s, A_n (which uses that $A_s = p(A)$ established below).

Then: $\underbrace{A_s - A'_s}_{\text{s.s. by Ex c)}} = \underbrace{A'_n - A_n}_{\text{nilpotent by Ex d)}} \implies A'_s = A_s,\ A'_n = A_n \implies$ uniqueness!

- By Chinese remainder thm, $\exists\, p(t) \in \mathbb{C}[t]$ s.t. $p(t) \equiv \lambda_i \mod (t - \lambda_i)^{\dim V_{\lambda_i}}$ for any $\lambda_i \in \mathbb{C}$ s.t. $V_{\lambda_i} \neq 0$. Then: $(A - \lambda_i)^{\dim V_{\lambda_i}} = 0$ on V_{λ_i}

Therefore, $\boxed{A_s = p(A)}$

- So far, we have established the unique decomposition of A into $A = A_s + A_n$ with A_s - s.s., A_n - nilp, $[A_s, A_n]$, and proved that both A_s, A_n are pol-s in A. Moreover, if $V_0 \neq 0$, then $p(t) \in t\,\mathbb{C}[t]$.

- Finally, note that $ad(A) = ad(A_s) + ad(A_n) : End(V) \to End(V)$.

Here, $ad(A_s)$ & $ad(A_n)$ - commute, $ad(A_s)$ - s.s, $ad(A_n)$ - nilpotent.

[**Exercise**: Check this!

By uniqueness of Jordan decomp $ad(A_s) = (ad(A))_s$, $ad(A_n) = (ad(A))_n$.

Moreover, as $Ad(A) A = 0$, can write $ad(A_s) = p(ad\,A)$ with $p \in t\,\mathbb{C}[t]$.

[**Exercise**: Prove part c) of Thm 4