Lecture # 13 - 14

- Last time :
  - semisimplicity through invariant forms
  - all classical Lie algebras are reductive
  - Killing form
  - Cartan's Criteria for solvability
  - Cartan's Criteria for semisimplicity
  - Jordan decomposition

- Finish the proof of parts (a,b) of Jordan decomposition.

- Properties of semisimple Lie algebras

As an immediate corollary of Cartan's criterion for semisimplicity, we have:

Lemma 1 : If $char(\mathbb{k})=0$, then a fin. dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{k}$ is semisimple iff $\mathfrak{g}\otimes_{\mathbb{k}}\bar{\mathbb{k}}$ is semisimple.

Another important result is that every ideal admits a complimentary ideal:

Lemma 2 : Let $\mathfrak{g}$ be a semisimple Lie algebra, $I\subseteq\mathfrak{g}$ - ideal. Then there is an ideal $J\subseteq\mathfrak{g}$ such that $\mathfrak{g}=I\oplus J$

► Let $I^\perp$ be the orthogonal complement of I w.r.t Killing form. As discussed last time: $I^\perp$ - ideal as well. We claim that $\mathfrak{g}=I\oplus I^\perp$. To this end, it suffices to verify $I\cap I^\perp=0$ (for dimension reasons). But $I\cap I'$ is an ideal of $\mathfrak{g}$ with zero Killing form ([Hwk 6, Problem 3b]) hence it is solvable by Cartan's criterion for solvability.
But $\mathfrak{g}$ being semisimple implies then that $I\cap I^\perp=0$ ∎

As an immediate corollary of the result above, we get.

Corollary 1 : A Lie algebra is semisimple iff it is a direct sum of simple Lie alg.

► $\Leftarrow$: Use Cartan's criterion on the nose

$\Rightarrow$: If $\mathfrak{g}$ is not simple, find an ideal of least possible dimension, apply Lemma 2, and proceed by induction. ∎

Corollary 2 : For semisimple $\mathfrak{g}$, we have $[\mathfrak{g},\mathfrak{g}]=\mathfrak{g}$

► Apply Corollary 1 and the equality $[\mathfrak{g},\mathfrak{g}]=\mathfrak{g}$ for simple $\mathfrak{g}$ ∎

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Proposition 1: Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \dots \oplus \mathfrak{g}_k$ be a semisimple Lie algebra, with all $\mathfrak{g}_i$ being simple. Then, any ideal $I \subseteq \mathfrak{g}$ is of the form
$$I = \bigoplus_{i \in S} \mathfrak{g}_i \quad \text{for some subset } S \subseteq \{1, \dots, k\}$$

▶ The proof is by induction on k. Base case ($k=1$) is obvious.
Let $\pi_k : \mathfrak{g} \to \mathfrak{g}_k$ be the natural projection. Consider $\pi_k(I)$ – ideal of $\mathfrak{g}_k$

Case 1: $\pi_k(I) = 0 \Rightarrow I \subseteq \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_{k-1}$, and we can use the induction hypothesis

Case 2: $\pi_k(I) = \mathfrak{g}_k$. But then $[\mathfrak{g}_k, I] = [\mathfrak{g}_k, \mathfrak{g}_k] = \mathfrak{g}_k \Rightarrow \mathfrak{g}_k \subseteq I$.
In this case, $I = I' \oplus \mathfrak{g}_k$ for a subspace $I' \subseteq \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_{k-1}$ which is easily seen to be an ideal of $\mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_{k-1}$. The result follows from the induction hypothesis. ▮

In particular, we note:

Corollary 3: The ideal J in Lemma 2 is unique

As another important corollary, we get:

Corollary 4: a) Any ideal in a semisimple Lie algebra is semisimple
b) Any quotient of a semisimple Lie algebra is semisimple.

Let $\mathrm{Der}(\mathfrak{g})$ be the Lie algebra of derivations of a Lie algebra $\mathfrak{g}$ (cf. [Hwk3, Prob#5])
We have a natural map $\mathfrak{g} \xrightarrow{ad} \mathrm{Der}(\mathfrak{g})$, with $\mathrm{Ker}(ad) = \mathfrak{z}(\mathfrak{g})$ – center of $\mathfrak{g}$, and image $ad(\mathfrak{g})$ is an ideal of $\mathrm{Der}(\mathfrak{g})$, due to ([Hwk 3, Problem 5d]).
$$\boxed{[d, ad(x)] = ad(d(x)) \quad \forall d \in \mathrm{Der}(\mathfrak{g}) \ \forall x \in \mathfrak{g}}$$

Proposition 2: If $\mathfrak{g}$ is semisimple, then $\mathfrak{g} = \mathrm{Der}(\mathfrak{g})$

▶ Consider the invariant symmetric bilinear form
$$K(a,b) := tr_{\mathfrak{g}}(ab) \quad \forall a, b \in \mathrm{Der}(\mathfrak{g}).$$
It's an extension of the Killing form on $\mathfrak{g} \overset{ad}{\hookrightarrow} \mathrm{Der}(\mathfrak{g})$. Let $I = \mathfrak{g}^\perp$ be the orthogonal complement of $\mathfrak{g}$ in $\mathrm{Der}(\mathfrak{g})$ under K. Then: I - ideal, $I \cap \mathfrak{g} = 0$, and $\mathrm{Der}(\mathfrak{g}) = I \oplus \mathfrak{g}$. We then have $[\mathfrak{g}, I] = 0$ (as both $ad(\mathfrak{g}) = \mathfrak{g}$ & I - ideals)
But then $\forall d \in I, \forall x \in \mathfrak{g}$: $0 = [d, ad(x)] = ad(d(x)) \Rightarrow d(x) \in \mathfrak{z}(\mathfrak{g}) \Rightarrow d(x) = 0 \ \forall x \Rightarrow d = 0$.
Therefore: $I = 0$ and so $\mathfrak{g} \simeq \mathrm{Der}(\mathfrak{g})$ ▮

• For the rest of today we shall discuss extensions of $\mathfrak{g}$-modules

Let $\mathfrak{g}$ be a Lie algebre, and V, W be $\mathfrak{g}$-modules. One natural question to study is to classify all "extensions of W by V", i.e. $\mathfrak{g}$-modules U that fit into

$$\boxed{0 \to V \to U \xrightarrow{p} W \to 0} \quad \leftarrow \text{short exact sequence of } \mathfrak{g}\text{-mod.}$$

In other words, U is endowed with a 2-step filtration so that $gr.\, U \simeq V \oplus W$.

To this end, let's first pick a splitting of the above sequence as vector spaces, i.e. an injective linear map $\imath : W \hookrightarrow U$ so that $p \circ \imath = id_W$. This gives rise to a vector space isomorphism

$$\boxed{\tilde{\imath} : \underset{\uparrow \text{v. space isom}}{V \oplus W \longrightarrow U}, \quad (v, w) \mapsto v + \imath(w)}$$

hence $\mathfrak{g} \curvearrowright U$ can be rewritten via $\mathfrak{g} \overset{\rho}{\curvearrowright} V \oplus W$. Explicitly, we have

$$\boxed{\rho(x)(v, w) = (\rho_V(x)v + a(x)w, \rho_W(x)w)}$$

where $a : \mathfrak{g} \to \mathrm{Hom}_{\Bbbk}(W, V)$ - linear map (Note: $a = 0 \Leftrightarrow \tilde{\imath}$ - $\mathfrak{g}$-module isomorphism)

Let's now see which conditions for a guarantee above to define a $\mathfrak{g}$-action.

$$\rho([x,y])(v,w) = ([x,y].v + a([x,y])w, [x,y].w)$$
$$[\rho(x), \rho(y)](v,w) = ([x,y].v + ([x, a(y)] + [a(x), y])w, [x,y].w)$$

they are equal $\Leftrightarrow a : \mathfrak{g} \to \mathrm{Hom}_{\Bbbk}(W, V)$ satisfies the Leibniz-rule

$$\boxed{a([x,y]) = [x, a(y)] + [a(x), y] = [x, a(y)] - [y, a(x)]}$$

Def 1: In general, for a $\mathfrak{g}$-module E, "1-cocycles of $\mathfrak{g}$ with values in E", denoted $Z^1(\mathfrak{g}, E)$, are all linear maps $a : \mathfrak{g} \to E$ satisfying similar equality:

$$a([x,y]) = x.a(y) - y.a(x)$$

Examples: $Z^1(\mathfrak{g}, \Bbbk) \simeq (\mathfrak{g}/[\mathfrak{g},\mathfrak{g}])^*$

$Z^1(\mathfrak{g}, \mathfrak{g}) \simeq \mathrm{Der}(\mathfrak{g})$.

Upshot of the above discussion is that $a : \mathfrak{g} \to \mathrm{Hom}_{\Bbbk}(W, V)$ defines a $\mathfrak{g}$-module iff $a \in Z^1(\mathfrak{g}, \mathrm{Hom}_{\Bbbk}(W,V))$. For such a, let U_a be the corresponding extension $0 \to V \to U_a \to W \to 0$. However, since $\imath$ was not unique, we need to decide when $U_a \simeq U_b$ for $a \neq b$.

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To address the last question (namely when $\mathcal{U}_a$ is isom. to $\mathcal{U}_b$) we consider isomorphisms $f: \mathcal{U}_a \to \mathcal{U}_b$ that preserve the structure of the short exact sequences, i.e. $gr.(f) = id_{V\oplus W}$. Such f take the following form:

$$\boxed{f(v,w) = (v + Aw, w)}$$

where $A: W \to V$ - linear map. Let's now see what the conditions on A are that guarantee f to be a $\mathfrak{g}$-module homom. To this end, we compute xf & fx

$$\left.\begin{aligned} xf(v,w) &= x(v+Aw, w) = (x.v + x.Aw + b(x)w,\ x.w) \\ fx(v,w) &= f(x.v + a(x)w,\ x.w) = (x.v + a(x)w + A(x.w),\ x.w) \end{aligned}\right\} \Downarrow$$

the equality $xf = fx$ holds iff

$$\boxed{[x, A] = a(x) - b(x)}$$

<u>So</u>: two maps $a, b: \mathfrak{g} \to \mathrm{Hom}_{\mathbb{k}}(W,V)$ determine isomorphic $\mathfrak{g}$-module extensions iff $\exists\, A: W \to V$ s.t. $[x, A] = \rho_V(x)A - A\rho_W(x)$ coincides with $a(x) - b(x)$ $\forall x \in \mathfrak{g}$.

<u>In particular</u> (setting $b=0$): $\mathcal{U}_a$ is the <u>trivial extension</u> (i.e $\simeq V\oplus W$) iff

$$\boxed{a(x) = [x, A] \text{ for some } A: W \to V}$$

this is precisely action of $x \in \mathfrak{g}$ on $A \in \mathrm{Hom}(W,V)$

<u>Def 2</u>: In general, for a $\mathfrak{g}$-module E, the linear map $a: \mathfrak{g} \to E$ given by $\boxed{a(x) = x.v \ \ \forall x \in \mathfrak{g}}$ and some fixed $v \in E$ is called the "<u>1-coboundary of v</u>" and denoted $a = dv$. The space of all <u>1-coboundaries</u> is denoted $B^1(\mathfrak{g}, E)$

Upshot of the above discussion is that

$$\boxed{\exists \text{ isomorphism } f: \mathcal{U}_a \xrightarrow{\sim} \mathcal{U}_b \text{ with } gr.(f) = id_{V\oplus W} \text{ iff } a - b \in B^1(\mathfrak{g}, E)}$$

<u>Def 3</u>: In general, for a $\mathfrak{g}$-module E, the quotient space

$$H^1(\mathfrak{g}, E) = Z^1(\mathfrak{g}, E) / B^1(\mathfrak{g}, E) \qquad (\leftarrow \text{it's obvious that } B^1 \subseteq Z^1)$$

is called the "<u>1st cohomology of $\mathfrak{g}$ with coefficients in E</u>".

<u>Therefore</u>: every extension $\mathcal{U}$ of W by V determines a class $[\mathcal{U}] \in H^1(\mathfrak{g}, \mathrm{Hom}_{\mathbb{k}}(W,V))$ and vice versa every class determines an extension. Moreover, $\mathcal{U}$ is trivial iff $[\mathcal{U}] = 0$.

first Ext group

<u>Def 4</u>: $\mathrm{Ext}^1(W,V) := H^1(\mathfrak{g}, \mathrm{Hom}_{\mathbb{k}}(W,V))$

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- After the above general discussions, we are ready to state today's key result:

Theorem 1 (Whitehead theorem): For a semisimple Lie algebra $\mathfrak{g}$ in $\mathrm{char}(\Bbbk)=0$, and any fin. dimensional $\mathfrak{g}$-module V, we have $H^1(\mathfrak{g}, V)=0$

Examples: a) For $V=\Bbbk$ – trivial $\mathfrak{g}$-module, we know $Z^1(\mathfrak{g},\Bbbk)=(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}])^*$

$$\begin{array}{c} Z^1(\mathfrak{g},\Bbbk) \\ \cup \\ B^1(\mathfrak{g},\Bbbk)=0 \end{array}$$

and we thus recover $\mathfrak{g}=[\mathfrak{g},\mathfrak{g}]$ from Corollary 2.

b) For $V=\mathfrak{g}$ with adjoint action, we know

$$\begin{array}{ccc} Z^1(\mathfrak{g},\mathfrak{g}) & = & \mathrm{Der}(\mathfrak{g}) \\ \cup & & \cup \\ B^1(\mathfrak{g},\mathfrak{g}) & = & \mathrm{ad}(\mathfrak{g}) \end{array}$$

and we thus recover $\mathfrak{g}\simeq \mathrm{Der}(\mathfrak{g})$ from Proposition 2.

The proof will proceed in several steps. We start with the following:

Lemma 3: Let $\mathfrak{g}$ be any Lie algebra, and E – its fin. dim. module, and $C\in U(\mathfrak{g})$ be a central element s.t. $C|_{\mathrm{triv}}=0$, $C|_E=\lambda\cdot \mathrm{id}_E$. Then $H^1(\mathfrak{g},E)=0$.

! Note that in above notations $H^1(\mathfrak{g},E)=H^1(\mathfrak{g},\mathrm{Hom}_{\Bbbk}(\Bbbk,E))=\mathrm{Ext}^1(\Bbbk,E)$

▶ We need to show that any extension

$$0\to E\to U\xrightarrow{p}\Bbbk\to 0$$

splits. To this end it suffices to find $u\in U$ (actually unique!) s.t. $p(u)=1$, $Cu=0$.
Indeed, C being central and $C|_E$ having no zero eigenvalues means that $E\not\ni \Bbbk u$, hence, $U\simeq E\oplus \Bbbk u$ and $p(u)=1$ guarantees splitting of sequence above.

To construct u, pick any $\tilde{u}\in U$ s.t. $p(\tilde{u})=1$. As C-central & $C|_{\Bbbk}=0$, we note that $p(C\tilde{u})=C1=0 \Rightarrow C\tilde{u}\in E$, in particular, $C(C\tilde{u})=\lambda\cdot C\tilde{u}$. We set $u:=\tilde{u}-\frac{1}{\lambda}C\tilde{u}$. Then $p(u)=p(\tilde{u})=1$, and $Cu=C\tilde{u}-\frac{1}{\lambda}\lambda C\tilde{u}=0$.
(uniqueness of u is also obvious) ■

To apply this Lemma to semisimple $\mathfrak{g}$, we need some natural construction of central elements in $U\mathfrak{g}$. In fact, we have already seen this for $\mathfrak{sl}_2$ before.

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Let $\{x_i\}$ be a basis of $\mathfrak{g}$ and $\{x^i\}$ be the dual basis under a non-degenerate invariant bilinear form (e.g. the Killing form).

Def 4: The Casimir element (determined by the form) is $C := \sum_i x_i x^i \in U(\mathfrak{g})$

Lemma 4: C does not depend on the choice of a basis $\{x_i\}$, and is central

▷ Identifying $\mathfrak{g}\otimes\mathfrak{g} \simeq \mathfrak{g}\otimes\mathfrak{g}^* = \mathrm{End}(\mathfrak{g})$ using the nondeg. form, we see that the tensor $\sum x_i \otimes x^i$ is independent of choice of basis as it corresponds to $\mathrm{id}_{\mathfrak{g}}$. Furthermore, this also shows that this tensor is $\mathrm{ad}(\mathfrak{g})$-invariant.
Finally, C is just the image of this tensor under $\mathfrak{g}$-module morphism $\mathfrak{g}\otimes\mathfrak{g} \xrightarrow{\text{mult}} U(\mathfrak{g})$, hence it's independent of basis and central ∎

Lemma 5: Let $\mathfrak{g}$ be a semisimple Lie algebra in $\mathrm{char}(\mathbb{k})=0$, and V be a nontrivial irreducible fin. dim. $\mathfrak{g}$-module. Then there is a central element $C\in U(\mathfrak{g})$ s.t. $C|_{\mathbb{k}}=0$, $C|_V \neq 0$.

▷ Consider the invariant symmetric bilinear form on $\mathfrak{g}$ from last time

$$B_V(x,y) = \mathrm{tr}_V(xy)$$

First, we note that $B_V \not\equiv 0$. Indeed, if B_V was zero, then $\rho(\mathfrak{g}) \subseteq \mathfrak{gl}(V)$ would be solvable by Cartan's criterion $\overset{\mathfrak{g}\text{-s.s.}}{\Longrightarrow} \rho(\mathfrak{g})=0 \Rightarrow V$ - trivial $\mathfrak{g}$-module $\Rightarrow$ ↯.
Next, if B_V was non-degenerate we would just take Casimir elt for it. But, in general, we proceed as follows. Let $I = \mathrm{Ker}(B_V)$ - ideal of $\mathfrak{g}$. Then we can decompose $\mathfrak{g} = I \oplus \mathfrak{g}'$ for some semisimple Lie algebra $\mathfrak{g}'$ by Prop. 1.
Then $B_V|_{\mathfrak{g}'}$ is nondegenerate, and let C be the Casimir elt of $B_V|_{\mathfrak{g}'}$, so that $C\in U(\mathfrak{g}') \subseteq U(\mathfrak{g})$. Then, C is not only central in $U(\mathfrak{g}')$ but also in $U(\mathfrak{g})$ as $[I,\mathfrak{g}']=0$.
Also: $\mathrm{tr}_V(C) = \sum_i B_V(x_i, x^i) = \dim(\mathfrak{g}')$, from which we find (using Schur's Lemma):
$C|_V = \frac{\dim(\mathfrak{g}')}{\dim V}\cdot \mathrm{id}_V$ - nonzero scalar operator. But $C|_{\mathbb{k}} \equiv 0$. ∎

Corollary 5: For any irreducible f.d. representation V of a semisimple Lie alg $\mathfrak{g}$ over $\mathbb{k}$ of $\mathrm{char}(\mathbb{k})=0$, we have $H^1(\mathfrak{g}, V)=0$

▷ If V - nontrivial, then it follows from Lemmas 3+5.
For $V=\mathbb{k}$, have $H^1(\mathfrak{g},\mathbb{k}) = (\mathfrak{g}/[\mathfrak{g},\mathfrak{g}])^* = 0$. Alternatively, if $0\to\mathbb{k}\to U\to\mathbb{k}\to 0$, then $\rho_U(x) = \begin{pmatrix} 0 & * \\ 0 & 0\end{pmatrix}$ - nilpotent $\Rightarrow \rho(\mathfrak{g})=0$ as $\mathfrak{g}$ - s.s. Lie alg. ∎

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Now we are ready to finish the proof of Whitehead's theorem.

Exercise: Verify that for a short exact sequence (s.e.s) of $\mathfrak{g}$-modules $0 \to V \to U \to W \to 0$, one obtains an exact sequence
$$H^1(\mathfrak{g}, V) \to H^1(\mathfrak{g}, U) \to H^1(\mathfrak{g}, W).$$

Proof of Whitehead theorem

By Corollary 5, $H^1(\mathfrak{g}, V) = 0$ for any irreducible f.d $\mathfrak{g}$-module V. Using Exercise above, and considering a Jordan-Holder filtration of any f. dim. $\mathfrak{g}$-module U, we obtain the desired vanishing $H^1(\mathfrak{g}, U) = 0$ ∎

As an immediate corollary, we obtain

Theorem 2: Every fin. dim. repr. of a semisimple Lie algebra $\mathfrak{g}$ over a field of char $= 0$ is completely reducible, i.e. $\simeq \bigoplus$ irreducibles.

By Theorem 1, we have $\mathrm{Ext}^1(W, V) = 0$. Hence, any short exact sequence
$$0 \to V \to U \to W \quad \text{splits!}$$
This implies the result.
(indeed if given U is not simple, find a submodule V inducing above s.e.s) split it as $U \simeq V \oplus W$ and repeat the argument for V, W!) ∎

As another important corollary of Theorem 1, we recover again:

Corollary 6: In char$(\mathbb{k}) = 0$, a reductive Lie algebra $\mathfrak{g}$ (uniquely) splits as a direct sum of abelian and semisimple Lie alg-s.

The adjoint action $\mathfrak{g} \curvearrowright \mathfrak{g}$ gives rise to $\underbrace{\mathfrak{g}_{ss} = \mathfrak{g}/\mathfrak{z}(\mathfrak{g})}_{\text{semisimple Lie alg.}} \curvearrowright \mathfrak{g}$, which fits into
$$0 \to \mathfrak{z}(\mathfrak{g}) \to \mathfrak{g} \to \mathfrak{g}_{ss} \to 0$$
By complete reducibility, we have $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{g}_{ss}$ (direct sum of ideals)
(uniqueness easily follows from $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$) ∎