Lecture #15

- Last time
  - complete reducibility of fin. dim $\mathfrak{g}$-modules for semisimple $\mathfrak{g}$ (and Casimir element)
- Address Survey's suggestions
- In the first part of today's class - give some hints on Hwk #6

- Semisimple elements and toral algebras

Def 1: a) An element $x \in \mathfrak{g}$ is called *semisimple* if $\mathrm{ad}\, x : \mathfrak{g} \to \mathfrak{g}$ is a semisimple operator
b) An element $x \in \mathfrak{g}$ is called *nilpotent* if $\mathrm{ad}\, x$ is a nilpotent operator

(**Exercise**: Show that for $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$ these definitions agree with the usual ones.

The following result generalizes Jordan decomposition to semisimple Lie algebras.

Proposition 1: Let $\mathfrak{g}$ be a semisimple complex Lie algebra. Then every element $x \in \mathfrak{g}$ admits a unique decomposition
$$x = x_s + x_n$$
with x_s - semisimple, x_n - nilpotent, and $[x_s, x_n] = 0$
Moreover, if $[x, y] = 0$ for some $y \in \mathfrak{g}$, then $[x_s, y] = [x_n, y] = 0$.

▶ Let us write $\mathfrak{g}$ as a direct sum of generalized eigenspaces for $\mathrm{ad}\, x$
$$\mathfrak{g} = \bigoplus_{\lambda \in \mathbb{C}} \mathfrak{g}_\lambda, \quad \mathfrak{g}_\lambda = \{ y \in \mathfrak{g} \mid (\mathrm{ad}\, x - \lambda)^n y = 0 \text{ for } n >> 0 \}$$

Lemma 1: $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subset \mathfrak{g}_{\lambda+\mu}$

▷ By Jacobi identity $(\mathrm{ad}\, x)[y, z] = [(\mathrm{ad}\, x) y, z] + [y, (\mathrm{ad}\, x) z]$
$$\Rightarrow (\mathrm{ad}\, x - \lambda - \mu)[y, z] = [(\mathrm{ad}\, x - \lambda) y, z] + [y, (\mathrm{ad}\, x - \lambda) z]$$
Thus: $(\mathrm{ad}\, x - \lambda - \mu)^N [y, z] = \sum_{k=0}^{N} \binom{N}{k} [(\mathrm{ad}\, x - \lambda)^k y, (\mathrm{ad}\, x - \mu)^{N-k} z]$

Hence, if $y \in \mathfrak{g}_\lambda$, $z \in \mathfrak{g}_\mu$, $N > \dim \mathfrak{g}_\lambda + \dim \mathfrak{g}_\mu$, each term in the right-hand side vanishes and so $[y, z] \in \mathfrak{g}_{\lambda+\mu}$. ○

Back to the proof of Prop 1, consider the Jordan decomposition in $\mathfrak{gl}(V)$: $\mathrm{ad}(x) = \mathrm{ad}(x)_s + \mathrm{ad}(x)_n$. By explicit construction: $\mathrm{ad}(x)_s|_{\mathfrak{g}_\lambda} = \lambda \cdot \mathrm{Id}$. But the above Lemma implies then that $\mathrm{ad}(x)_s \in \mathrm{Der}(\mathfrak{g})$. However, $\mathrm{Der}(\mathfrak{g}) = \mathfrak{g}$ by [Lecture 13, Prop 2] as $\mathfrak{g}$-semisimple, hence $\exists\, x_s \in \mathfrak{g}$ s.t. $\mathrm{ad}(x)_s = \mathrm{ad}(x_s)$. Take $x_n = x - x_s$. This proves existence. The uniqueness follows from $\mathfrak{g} \hookrightarrow \mathfrak{gl}(\mathfrak{g})$ and uniqueness in $\mathfrak{gl}(V)$.
Finally, if $[x, y] = 0$, then $\mathrm{ad}\, y$ preserves each $\mathfrak{g}_\lambda \Rightarrow \mathrm{ad}([x_s, y]) = 0 \Rightarrow [x_s, y] = 0 \Rightarrow [x_n, y] = 0$.

Lecture #14

As an immediate corollary, we get:

Corollary 1: In any semisimple complex Lie algebra $\mathfrak{g}$, there exist non-zero semisimple elts

▶ If not, then by Proposition 1, every $x \in \mathfrak{g}$ is nilpotent
But then $\mathfrak{g}$ - nilpotent by Engel's theorem, hence solvable, contradiction. ∎

But in what follows, we shall need a whole family of commuting semisimple elts.

Def 2: A subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ is called *toral* if it's commutative and consists of semisimple elements

Example: For $\mathfrak{g} = \mathfrak{sl}_n$, any subspace of diagonal traceless matrices forms a toral subalg.

Proposition 2: Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{h} \subseteq \mathfrak{g}$ - toral subalg, and $(\cdot,\cdot)$ be a non-degenerate symmetric bilinear form on $\mathfrak{g}$ (e.g. Killing). Then:

a) $\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha$, $\mathfrak{g}_\alpha$ - common eigenspace for all $\{ad(h) \mid h \in \mathfrak{h}\}$, i.e.
$$\mathfrak{g}_\alpha = \{y \in \mathfrak{g} \mid [h,y] = \alpha(h)\cdot y \ \ \forall h \in \mathfrak{h}\}$$
and $\mathfrak{h} \subseteq \mathfrak{g}_0$.

b) $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$

c) if $\alpha+\beta \neq 0$, then $(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$

d) $\forall \alpha$, the form $(\cdot,\cdot)$ gives rise to a non-degenerate pairing $\mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha} \to \mathbb{C}$

▶ a) Is just the statement that a commuting family of diagonalizable operators can be simultaneously diagonalized (basic linear algebra)

b) Follows from Lemma 1 (actually, it's a very special case of it).

c) Pick any α, β and $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_\beta$. Then for any $h \in \mathfrak{h}$:
$$\left.\begin{array}{l} 0 = ([h,x], y) + (x, [h,y]) = (\alpha(h)+\beta(h))\cdot(x,y) \\ \text{If } \alpha+\beta \neq 0 \Rightarrow \exists\, h \in \mathfrak{h} \text{ s.t. } \alpha(h)+\beta(h) \neq 0 \end{array}\right\} \Rightarrow (x,y) = 0$$

d) Follows from c) and nondegeneracy of $(\cdot,\cdot)$ ∎

Lecture #14

As an immediate corollary, we get:

Corollary 2: a) $\mathfrak{g}_0$ is a reductive subalgebra

b) if $x \in \mathfrak{g}_0$, then $x_s, x_n \in \mathfrak{g}_0$

▶ a) Consider $\mathrm{ad}: \mathfrak{g}_0 \curvearrowright \mathfrak{g}$. The corresponding trace form $(x_1, x_2) = \mathrm{tr}_{\mathfrak{g}}(\mathrm{ad}\, x_1 \circ \mathrm{ad}\, x_2)$ on $\mathfrak{g}_0$ is precisely $K^{\mathfrak{g}}_{|\mathfrak{g}_0 \times \mathfrak{g}_0}$, which is non-degenerate by Prop 2d). Hence, $\mathfrak{g}_0$ is reductive by [Lecture 12, Lemma 2].

b) if $x \in \mathfrak{g}_0 \Rightarrow [h, x] = 0 \ \forall h \in \mathfrak{h} \overset{\text{Prop 1}}{\Longrightarrow} [x_s, h] = 0 = [x_n, h] \ \forall h \in \mathfrak{h} \Rightarrow x_s, x_n \in \mathfrak{g}_0$ ■

- Cartan subalgebras

One would like to work with maximal toral algebras (for the reason that will become clear next time).

Def 3: Let $\mathfrak{g}$ be a complex semisimple Lie algebra. A Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ is a toral subalgebra $\mathfrak{h} \subset \mathfrak{g}$ s.t. $\mathfrak{g}_0 = \mathfrak{h}$

Example: For $\mathfrak{g} = \mathfrak{sl}_n$, take $\mathfrak{h} = \{$all traceless diagonal matrices$\}$. Then: $\mathfrak{h}$ – Cartan subalg. of $\mathfrak{g}$

Existence of such subalgebras follows from the following simple result.

Proposition 3: Let $\mathfrak{h} \subseteq \mathfrak{g}$ be a maximal toral subalgebra (i.e. not properly contained in another toral subalgebra). Then $\mathfrak{h}$ is a Cartan subalgebra.

▶ Consider the decomposition $\mathfrak{g} = \oplus \mathfrak{g}_\alpha$ from Prop 1. As $\mathfrak{h} \subseteq \mathfrak{g}_0$, it suffices to show that $\mathfrak{g}_0$ is toral.

Let $x \in \mathfrak{g}_0$, then $x_s, x_n \in \mathfrak{g}_0$ by Corollary 2b). By maximality of $\mathfrak{h}$, we must have $x_s \in \mathfrak{h}$. But then $\mathrm{ad}(x_s)_{|\mathfrak{g}_0} = 0$ and so $\mathrm{ad}(x)_{|\mathfrak{g}_0} = \mathrm{ad}(x_n)_{|\mathfrak{g}_0}$ – nilpotent $\forall x \in \mathfrak{g}_0 \overset{\text{Engel's thm}}{\Longrightarrow} \mathfrak{g}_0$ is nilpotent.

However, by Corollary 2a), $\mathfrak{g}_0$ – reductive. Hence: $\mathfrak{g}_0$ – abelian.

To prove that any $x \in \mathfrak{g}_0$ is semisimple, need to show $x_n = 0$. Then as $\mathfrak{g}_0$ is abelian: $\forall y \in \mathfrak{g}_0 \ [x_n, y] = 0 \Rightarrow \underbrace{\mathrm{ad}\, x_n}_{\text{nilpotent}} \circ \underbrace{\mathrm{ad}\, y}_{\text{commuting operator with } \mathrm{ad}(x_n)} : \mathfrak{g} \to \mathfrak{g}$ is nilpotent $\Rightarrow \mathrm{tr}_{\mathfrak{g}}(\mathrm{ad}\, x_n \circ \mathrm{ad}\, y) = 0 \ \forall y \in \mathfrak{g}_0$

But this form is nondegenerate on $\mathfrak{g}_0$, hence, $x_n = 0$. ■

We will see later that all Cartan subalgebras of $\mathfrak{g}$ are conjugate under $\mathrm{Aut}(\mathfrak{g})$. Thus, they all have the same dimension, called the rank of $\mathfrak{g}$: $\boxed{\dim(\mathfrak{h}) = \mathrm{rank}(\mathfrak{g})}$