Lecture #16

- Last time:
  - notion of semisimple & nilpotent el-s in a Lie algebra
  - abstract Jordan decomposition for s.s. Lie algebras
  - toral subalgebras and decomposition $\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha$
  - Cartan subalgebras (definition and example for $\mathfrak{sl}_n$)
- Finish last page from previous notes (existence of Cartan subalg; notion of $\mathrm{rank}(\mathfrak{g})$)
- For the rest of today, we shall focus on the root decomposition and root systems (the core in the theory of semisimple Lie algebras)

We fix a complex semisimple Lie algebra $\mathfrak{g}$, and a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$. In this case, Prop. 2 of Lecture #15 provides:

Theorem 1: For $\mathfrak{h} \subseteq \mathfrak{g}$ as above, we have:

a) root decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha, \quad \mathfrak{g}_\alpha := \{x \in \mathfrak{g} \mid [h, x] = \alpha(h) \cdot x \quad \forall h \in \mathfrak{h}\}$$

b) $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$ (we let $\mathfrak{g}_0 := \mathfrak{h}$)

or we could take any other nondeg. invariant symm. bilinear form on $\mathfrak{g}$

c) if $\alpha + \beta \neq 0$, then $\mathfrak{g}_\alpha$ & $\mathfrak{g}_\beta$ are orthogonal w.r.t. Killing form K

d) $\forall \alpha$ s.t. $\mathfrak{g}_\alpha \neq 0$, the Killing form gives a non-deg. pairing $\mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha} \to \mathbb{C}$. In particular (taking $\alpha = 0$), restriction $K|_{\mathfrak{h} \times \mathfrak{h}}$ is non-degenerate.

Def 1: a) $R := \{\alpha \in \mathfrak{h}^* \setminus \{0\} \mid \mathfrak{g}_\alpha \neq 0\}$ – root system of $\mathfrak{g}$

b) $\mathfrak{g}_\alpha$ ($\alpha \in R$) – root subspaces.

Proposition 1: Let $\mathfrak{g}_1, \ldots, \mathfrak{g}_n$ be simple Lie algebras, and set $\mathfrak{g} := \bigoplus_{i=1}^n \mathfrak{g}_i$.

a) Let $\mathfrak{h}_i \subseteq \mathfrak{g}_i$ be Cartan subalgebras and $R_i \subseteq \mathfrak{h}_i^*$ be the corresponding root systems. Then $\mathfrak{h} := \bigoplus_{i=1}^n \mathfrak{h}_i$ is a Cartan subalgebra of $\mathfrak{g}$, and the corresp. root system is $R = \coprod R_i$ (union of R_i's)

b) Any Cartan subalgebra in $\mathfrak{g}$ is of the form $\mathfrak{h} = \bigoplus_{i=1}^n \mathfrak{h}_i$, with $\mathfrak{h}_i \subseteq \mathfrak{g}_i$ - Cartan

▶ a) Obvious.

b) Set $\mathfrak{h}_i := \pi_i(\mathfrak{h})$, where $\pi_i : \mathfrak{g} = \bigoplus_{k=1}^n \mathfrak{g}_k \xrightarrow{\text{projection}} \mathfrak{g}_i$. Then $[h, x] = [\pi_i(h), x]$ $\forall x \in \mathfrak{g}_i$. Thus, $\mathfrak{h}_i$ - Cartan subalg. of $\mathfrak{g}_i$. Therefore, $\mathfrak{h} \underset{\uparrow}{\subseteq} \bigoplus_{i=1}^n \mathfrak{h}_i$. But by a), $\oplus \mathfrak{h}_i$ - toral. Hence, as $\mathfrak{h}$ - Cartan, we must have the equality.

Example: Let $\mathfrak{g} = sl_n(\mathbb{C})$, $\mathfrak{h} = \{$diagonal trace$=0$ matrices$\}$ – Cartan subalgebra of $\mathfrak{g}$.
Consider $e_1, \dots, e_n \in \mathfrak{h}^*$ defined via $e_i : \begin{pmatrix} x_1 & & 0 \\ & x_2 & \\ 0 & & x_n \end{pmatrix} \mapsto x_i$.
Note that $x_1 + \dots + x_n = 0$ implies that $e_1 + \dots + e_n = 0$. Thus, we can identify

$$\mathfrak{h}^* = \bigoplus \mathbb{C} \cdot e_i \big/ \mathbb{C}(e_1 + \dots + e_n)$$

We also have $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{1 \le i \ne j \le n} \mathbb{C} E_{ij}$, and $[\begin{pmatrix} x_1 & & 0 \\ 0 & & x_n \end{pmatrix}, E_{ij}] = (x_i - x_j) E_{ij}$. Hence, the root system R. of $sl_n(\mathbb{C})$ consists of

$$\boxed{R = \{ e_i - e_j \mid i \ne j \} \text{ and } \mathfrak{g}_{e_i - e_j} = \mathbb{C} \cdot E_{ij}}$$

In particular, $|R| = n(n-1)$

Let $\mathfrak{h} \subseteq \mathfrak{g}$ be a Cartan subalgebra as above. We also fix $(\cdot,\cdot) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$, a nondegenerate invariant symmetric bilinear form on $\mathfrak{g}$, e.g. the Killing form. As $(\cdot,\cdot)|_{\mathfrak{h}\times\mathfrak{h}}$ is non-degenerate, it defines an isomorphism of v. spaces $\mathfrak{h} \xrightarrow{\sim} \mathfrak{h}^*$, $h \mapsto (h, -)$.
The inverse isomorphism $\mathfrak{h}^* \xrightarrow{\sim} \mathfrak{h}$ shall be denoted $\alpha \mapsto H_\alpha$. Moreover, we also get a non-degenerate bilinear form on $\mathfrak{h}^*$, also denoted $(\cdot,\cdot)$, so that

$$\boxed{(\alpha, \beta) = \alpha(H_\beta) = (H_\alpha, H_\beta) \qquad \forall \alpha, \beta \in \mathfrak{h}^*}$$

Lemma 1: For any $\alpha \in R$ and any $e \in \mathfrak{g}_\alpha, f \in \mathfrak{g}_{-\alpha}$, we have $[e,f] = (e,f) \cdot H_\alpha$

▶ As $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subseteq \mathfrak{g}_0 = \mathfrak{h}$, the Lie bracket $[e,f] \in \mathfrak{h}$. As the pairing $(\cdot,\cdot)|_{\mathfrak{h}\times\mathfrak{h}}$ is nondeg., to prove the above equality it suffices to show that pairings with any $h \in \mathfrak{h}$ coincide.
But: $([e,f], h) \overset{\text{inv. pairing}}{=\!=\!=} (e, [f,h]) = \alpha(h) \cdot (e,f) = ((e,f) \cdot H_\alpha, h)$ ∎

Lemma 2: a) $\forall \alpha \in R$, $(\alpha, \alpha) = (H_\alpha, H_\alpha) \ne 0$

b) Let $\alpha \in R$ and $e \in \mathfrak{g}_\alpha$, $f \in \mathfrak{g}_{-\alpha}$ be such that $(e,f) = \frac{2}{(\alpha,\alpha)}$, and set

$$\boxed{h_\alpha := \frac{2 H_\alpha}{(\alpha,\alpha)}}$$

Then $\{e, f, h_\alpha\}$ satisfy the commutation relations of sl_2.

c) The above element $h_\alpha \in \mathfrak{h}$ is independent of the choice of $(\cdot,\cdot)$

This result provides a very important tool in the study of $\mathfrak{g}$. Namely, we can consider $\mathfrak{g}$ as a module over the corresponding sl_2-subalgebra, denoted

$$\boxed{sl(2,\mathbb{C})_\alpha, \quad \alpha \in R.}$$

Lecture #16

Proof of Lemma 2

a) Assume the contradiction, i.e. $(\alpha,\alpha)=0$ for some $\alpha\in R$. Then $\alpha(H_\alpha)=0$.
As $(\cdot,\cdot)|_{\mathfrak{g}_\alpha\times\mathfrak{g}_{-\alpha}}$ is non-degenerate, we can pick $e\in\mathfrak{g}_\alpha$, $f\in\mathfrak{g}_{-\alpha}$ s.t. $(e,f)\neq 0$.
Set $h:=[e,f]$ which equals $h=(e,f)\cdot H_\alpha$ by Lemma 1. We consider

Lie subalgebra $\mathfrak{a}\subseteq\mathfrak{g}$ generated by e,f,h.

As $\alpha(H_\alpha)=0\Rightarrow\alpha(h)=0$, we have $[h,e]=0=[h,f]\Rightarrow$ $\mathfrak{a}$ - solvable.
Then, by Lie's Theorem, there is a basis of $\mathfrak{g}$ s.t. $ad(e)$, $ad(f)$, $ad(h)$ are represented by upper-Δ matrices. Moreover, $ad(h)=[ad(e),ad(f)]$ must be strictly upper-Δ, hence, nilpotent. But $\mathfrak{h}\ni h$ - semisimple $\Rightarrow ad(h)$ - semisimple

Hence: $ad(h)=0\xRightarrow{\mathfrak{g}\text{-s.s.}} h=0\Rightarrow$ Contradiction as $h=\underbrace{(e,f)}_{\neq 0}\cdot H_\alpha$.

b) Obvious, using Lemma 1 (check it!)

c) By Prop 1, it suffices to check for simple $\mathfrak{g}$. But in this case, any nondeg. inv. symm. bilinear form is a multiple of the Killing form and the result easily follows (check it!)

$\blacksquare$

Lemma 3: For $\alpha\in R$, let $\mathfrak{a}_\alpha:=\mathbb{C}\cdot H_\alpha\oplus\bigoplus_{k\in\mathbb{Z}\setminus\{0\}}\mathfrak{g}_{k\alpha}\subseteq\mathfrak{g}$. Then $\mathfrak{a}_\alpha$ - Lie subalg. of $\mathfrak{g}$

As $[\mathfrak{g}_\alpha,\mathfrak{g}_\beta]\subseteq\mathfrak{g}_{\alpha+\beta}$, the only nontrivial check to do is that $[\mathfrak{g}_{k\alpha},\mathfrak{g}_{-k\alpha}]\subseteq\mathbb{C}\cdot H_\alpha$.
Pick any $e\in\mathfrak{g}_{k\alpha}$, $f\in\mathfrak{g}_{-k\alpha}$. Then:

$[e,f]=(e,f)\cdot H_{k\alpha}=(e,f)\cdot k\cdot H_\alpha\in\mathbb{C}\cdot H_\alpha$

$\blacksquare$

Lemma 4: Let $\alpha\in R$ and recall the $\mathfrak{sl}_2(2,\mathbb{C})_\alpha$ generated by $e\in\mathfrak{g}_\alpha$, $f\in\mathfrak{g}_{-\alpha}$, and h_α.

a) $\mathfrak{a}_\alpha$ is an irreducible $\mathfrak{sl}(2,\mathbb{C})_\alpha$ - module w.r.t. adjoint action.

b) $\dim\mathfrak{g}_\alpha=1$.

c) $\mathfrak{g}_{k\alpha}=0$ $\forall k\in\mathbb{Z}_{\geq 2}$.

The proof of this result shall crucially use the repr. theory of $\mathfrak{sl}_2$, as discussed back in Lecture 7, see Theorem 1 there.

Lecture #16

Proof of Lemma 4

a) As $[\mathfrak{g}_\alpha, \mathfrak{g}_{k\alpha}] \subseteq \mathfrak{g}_{(k+1)\alpha}$ and $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subseteq \mathbb{C} h_\alpha$ (Lemma 3), we see that $\mathfrak{a}_\alpha$ is $ad(e)$-stable.

Similarly, $\mathfrak{a}_\alpha$ is also $ad(f)$-invariant/stable. This shows that $\mathfrak{sl}(2,\mathbb{C})_\alpha \curvearrowright \mathfrak{a}_\alpha$.

Note that $\alpha(h_\alpha) = \frac{2}{(\alpha,\alpha)} \cdot \alpha(H_\alpha) = 2 \Rightarrow k\alpha(h_\alpha) = 2k$. Therefore, w.r.t. $ad(h_\alpha)$,

we have the decomposition $\boxed{\mathfrak{a}_\alpha = \bigoplus_{k\in\mathbb{Z}} \mathfrak{a}_\alpha[2k], \text{ with } \mathfrak{a}_\alpha[2k] = \begin{cases} \mathfrak{g}_{k\alpha}, & k\neq 0 \\ \mathbb{C}h_\alpha, & k=0. \end{cases}}$

As $\mathfrak{a}_\alpha[1]=0$ and $\dim \mathfrak{a}_\alpha[0]=1$, the repr. theory of $\mathfrak{sl}_2$ implies that $\mathfrak{a}_\alpha$ is an irreducible $\mathfrak{sl}(2,\mathbb{C})_\alpha$-module.

b) It immediately follows from a) that all $\mathfrak{a}_\alpha[2k]$ are 1-dim or zero, hence, $\dim \mathfrak{g}_\alpha = 1$.

c) As $[e,e]=0$ and $\mathfrak{g}_\alpha = \mathbb{C}\cdot e$ (by part b)), we have $ad(e): \mathfrak{g}_\alpha \to \mathfrak{g}_{2\alpha}$ is zero.

But again appealing to $\mathfrak{sl}_2$-theory, we see that the above implies $\mathfrak{g}_{k\alpha}=0 \ \forall k\geq 2$ ∎

Now we are ready to prove the main result about s.s. Lie algebras.

Theorem 2: Let $\mathfrak{g}$ be a complex semisimple Lie algebra with Cartan subalg. $\mathfrak{h}$ and root decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in R} \mathfrak{g}_\alpha$. Let $(\cdot,\cdot)$ be a non-degenerate invariant symmetric bilinear form on $\mathfrak{g}$. Then:

1) R spans $\mathfrak{h}^*$ as a vector space and $\{h_\alpha \mid \alpha\in R\}$ span $\mathfrak{h}$ as a v. space

2) For each $\alpha\in R$, the root subspace $\mathfrak{g}_\alpha$ is 1-dimensional

3) For any $\alpha,\beta\in R$, the number $a_{\alpha\beta} := \beta(h_\alpha) = \frac{2(\alpha,\beta)}{(\alpha,\alpha)}$ is an integer

4) For any $\alpha\in R$, define the
reflection operator $s_\alpha: \mathfrak{h}^* \to \mathfrak{h}^*$ via $\lambda \mapsto \lambda - \lambda(h_\alpha)\alpha = \lambda - \frac{2(\lambda,\alpha)}{(\alpha,\alpha)}\alpha$

Then for any roots $\alpha,\beta\in R$, $s_\alpha(\beta)$ is also a root.

5) For any root $\alpha\in R$, the only multiples of α which are also roots are $\pm\alpha$

6) For roots α and $\beta\neq\pm\alpha$, the subspace

$$V_{\alpha,\beta} := \bigoplus_{k\in\mathbb{Z}} \mathfrak{g}_{\beta+k\alpha}$$

is an irreducible $\mathfrak{sl}(2,\mathbb{C})_\alpha$-module

7) If $\alpha,\beta\in R$ are such that $\alpha+\beta\in R$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$

Lecture #16

Proof of Theorem 2

1) Assuming $h \in \mathfrak{h}$ is such that $\alpha(h)=0 \ \forall \alpha \in \mathfrak{h}^*$, we get $ad(h)=0$ due to the root decomposition. But $\mathfrak{g}$ - s.s. $\Rightarrow Z(\mathfrak{g})=0 \Rightarrow h=0$. This implies both results.

2) This is Lemma 4 b)

3) Considering the entire $\mathfrak{g}$ as a module over $\mathfrak{sl}(2,\mathbb{C})_\alpha$, we see that the weight of $\mathfrak{g}_\beta$ equals $\beta(h_\alpha) = \frac{2(\alpha,\beta)}{(\alpha,\alpha)}$. But from the $\mathfrak{sl}_2$-theory, we know that weights of fin. dim. modules are always integer. Hence, $a_{\alpha\beta} := \beta(h_\alpha) \in \mathbb{Z}$.

4) By part 3), know $\beta(h_\alpha) \in \mathbb{Z}$.

- If $\beta(h_\alpha)=0 \Rightarrow s_\alpha(\beta)=\beta \in R$.
- If $\beta(h_\alpha)=n \in \mathbb{Z}_{>0}$, then the root subspace $\mathfrak{g}_\beta$ has weight n w.r.t. $\mathfrak{sl}(2,\mathbb{C})_\alpha$. But evoking $\mathfrak{sl}_2$-theory (see [Lecture 9, Corollary 1]), we know that

$$ad(f)^n : \underbrace{\mathfrak{g}[n]}_{\substack{\text{weight } n \\ \text{subspace}}} \xrightarrow{\sim} \underbrace{\mathfrak{g}[-n]}_{\substack{\text{weight } -n \\ \text{subspace}}}$$

Therefore, $\forall x \in \mathfrak{g}_\beta \setminus \{0\}$ we have $ad(f)^n x \in \mathfrak{g}_{\underbrace{\beta - n\alpha}_{= s_\alpha(\beta)}}$ is nonzero! So $s_\alpha(\beta) \in R$

- If $\beta(h_\alpha)=-n \in \mathbb{Z}_{<0}$, use the same argument with $ad(e)^n$ instead.

5) Let $\alpha \in R$ and assume $\beta = c\cdot\alpha \in R$ for some $c \in \mathbb{C}$.

By part 3):
$$\left.\begin{array}{l} \frac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z} \Rightarrow 2c \in \mathbb{Z} \Rightarrow c \in \frac{1}{2}\mathbb{Z} \\ \frac{2(\beta,\alpha)}{(\beta,\beta)} \in \mathbb{Z} \Rightarrow \qquad \frac{1}{c} \in \frac{1}{2}\mathbb{Z} \end{array}\right\} \Rightarrow c \in \{\pm 1, \pm 2, \pm 1/2\}$$

Interchanging α & β, we can assume $c \in \{\pm 1, \pm 2\}$. But we proved $\mathfrak{g}_{\pm 2\alpha}=0$ in Lemma 4c). $\Rightarrow \pm 2\alpha \notin R$. This shows that $c=\pm 1$.

6) $V_{\alpha,\beta}$ is clearly an $\mathfrak{sl}(2,\mathbb{C})_\alpha$-module, and as $(\alpha,\alpha)\neq 0$, each $\mathfrak{g}_{\beta+k\alpha}$ is in different weight component. As $\dim(\mathfrak{g}_{\beta+k\alpha}) \leq 1$ (Lemma 4b), we conclude $V_{\alpha,\beta}$ is irreducible.

7) As $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$ and $\dim(\mathfrak{g}_\alpha)=\dim(\mathfrak{g}_\beta)=\dim(\mathfrak{g}_{\alpha+\beta})=1$, we need to show $[e_\alpha, e_\beta]\neq 0$ for nonzero $e_\alpha \in \mathfrak{g}_\alpha$, $e_\beta \in \mathfrak{g}_\beta$. But this follows from 6) and $\mathfrak{sl}_2$-theory. Indeed, $V_{\alpha,\beta}$ - irreducible $\Rightarrow ad(e_\alpha): \mathfrak{g}_\beta \hookrightarrow \mathfrak{g}_{\beta+\alpha}$ as long as $\beta, \beta+\alpha \in R$