Lecture # 17

- Last time
  - root systems and root subspaces
  - $\mathfrak{sl}_2(2,\mathbb{C})_\alpha \subseteq \mathfrak{g}$ for every root α
  - concluded with $\mathfrak{a}_\alpha := \mathbb{C}H_\alpha \oplus \bigoplus_{k\in\mathbb{Z}\setminus 0} \mathfrak{g}_{k\alpha}$ being irreducible $\mathfrak{sl}(2,\mathbb{C})_\alpha$-module.

Remark: For the proof of the last result as well as for the upcoming results, let's recall the basic stuff about $\mathfrak{sl}_2$ fin. dim. representations:

1) every fin. dim. repr-n is completely reducible, i.e. $\simeq \oplus$ irreducibles.
2) every irreducible fin. dim-l $\mathfrak{sl}_2$-module is $\simeq V_n$ for some n
3) the action of $h\in\mathfrak{sl}_2$ on V_n is diagonalizable, with eigenvalues $n, n-2, n-4, \ldots, -n+2, -n$, each of multiplicity 1

As an important consequence of 1)–3), we see that

$$\boxed{\#\text{ irreducible components in } \mathfrak{sl}_2\text{-module } V = \dim V(0) + \dim V(1)},$$

where $V(n)$ denotes the eigenspace of $h: V \circlearrowleft$ of eigenvalue n.
Furthermore, we will also need today that

$$\boxed{e^n:\ V(-n) \xrightarrow{\sim} V(n)\ ,\quad f^n:\ V(n)\xrightarrow{\sim} V(-n)\quad \forall n\in\mathbb{Z}_{>0}}$$

- Finish the proof of Lemma 4(b,c) from last time.
- Theorem 2 + its proof from last time

Corollary 1: Let $\mathfrak{h}_\mathbb{R}$ be the $\mathbb{R}$-span of $\{h_\alpha\}_{\alpha\in R}$. Then the restriction of the Killing form to $\mathfrak{h}_\mathbb{R}$ is positive definite and $\mathfrak{h}\simeq \mathfrak{h}_\mathbb{R}\oplus i\mathfrak{h}_\mathbb{R}$.

▷ For any $\alpha,\beta\in R$, we have $(h_\alpha, h_\beta) = \mathrm{tr}(\mathrm{ad}(h_\alpha)\mathrm{ad}(h_\beta)) = \sum_{\gamma\in R}\gamma(h_\alpha)\gamma(h_\beta)\in\mathbb{Z}$ by Thm 2(3).
For any $h=\sum_{\alpha\in R}\underbrace{c_\alpha}_{\in\mathbb{R}} h_\alpha\in\mathfrak{h}_\mathbb{R}$, have $\gamma(h)=\sum_{\alpha\in R} c_\alpha\cdot\underbrace{\gamma(h_\alpha)}_{\in\mathbb{Z}}\in\mathbb{R}\quad\forall\gamma\in R$. Hence, $(h,h)=\sum_{\gamma\in R}\gamma(h)^2\geq 0$ with the equality iff $h=0$. Thus, the restriction of the Killing form to $\mathfrak{h}_\mathbb{R}$ is <u>positive definite</u>, hence on $i\mathfrak{h}_\mathbb{R}$ its <u>negative definite</u>.
The above implies $\mathfrak{h}_\mathbb{R}\cap i\mathfrak{h}_\mathbb{R}=0 \Rightarrow \dim_\mathbb{R}\mathfrak{h}_\mathbb{R}\leq\dim_\mathbb{C}\mathfrak{h}_\mathbb{C}$. But since $\{h_\alpha\}_{\alpha\in R}$ generate $\mathfrak{h}$ over $\mathbb{C}$, we also have $\dim_\mathbb{R}\mathfrak{h}_\mathbb{R}\geq\dim_\mathbb{C}\mathfrak{h}_\mathbb{C}$, hence equality $\Rightarrow \mathfrak{h}\simeq\mathfrak{h}_\mathbb{R}\oplus i\mathfrak{h}_\mathbb{R}$ ▨

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- Strongly regular elements

Recall that last time we made the following observation: for $\mathfrak{g}=\mathfrak{sl}_n(\mathbb{C})$, $\mathfrak{h}\subseteq\mathfrak{g}$ – Cartan subalg. of diagonal matrices, we have $\mathfrak{h}=$ centralizer of $X=\begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_n\end{pmatrix}=\{Y\in\mathfrak{g}\mid [Y,X]=0\}$, whenever $x_1, x_2, \dots, x_n$ – pairwise distinct.

Hence, it is natural to ask if we can always realize Cartan subalgebras in such a way. The answer is Yes and is based on the following definition.

Def 1: For any element $x\in\mathfrak{g}$, define the nullity of x by

$$\boxed{n(x) := \text{multiplicity of } 0 \text{ as a generalized eigenvalue of } \mathrm{ad}(x):\mathfrak{g}\to\mathfrak{g}}$$

Def 2: For a Lie algebra $\mathfrak{g}$, its rank is defined by

$$\boxed{\mathrm{rank}(\mathfrak{g}) := \min_{x\in\mathfrak{g}} n(x)}$$

An element $x\in\mathfrak{g}$ is strongly regular if $n(x)=\mathrm{rank}(\mathfrak{g})$.

Note that $\mathrm{rank}(\mathfrak{g})\geq 1$ always, as $n(x)\geq 1$ $\forall x\in\mathfrak{g}$ (due to $[x,x]=0$)

Exercise: For $\mathfrak{g}=\mathfrak{sl}_n$, verify that $x\in\mathfrak{g}$ is strongly regular $\Longleftrightarrow$ eigenvalues of x are pairwise distinct

Lemma 1: In any finite-dimensional complex Lie algebra $\mathfrak{g}$, the set $\mathfrak{g}^{sr}$ of strongly regular elements is connected, dense, and open in $\mathfrak{g}$.

► For $x\in\mathfrak{g}$, consider the characteristic polynomial of $\mathrm{ad}(x):\mathfrak{g}\to\mathfrak{g}$

$$p_x(t)=\det(\mathrm{ad}\,x - t) = a_N(x)t^N + a_{N-1}(x)t^{N-1}+\dots+a_0(x)$$

where $N=\dim\mathfrak{g}$ and each $a_i(x)$ is a polynomial function on $\mathfrak{g}$.

By the very definition, $n(x)=\min\{k\mid a_k(x)\neq 0\}$. Thus, $x\in\mathfrak{g}^{sr}$ iff $a_r(x)\neq 0$ where $r:=\mathrm{rank}(\mathfrak{g})$, while $a_0(y)=a_1(y)=\dots=a_{r-1}(y)=0$ $\forall y\in\mathfrak{g}$. As $r=\mathrm{rank}(\mathfrak{g})$, the polynomial a_r does not vanish on $\mathfrak{g}$. Therefore, we just need to show that:

$$\boxed{\{x\in\mathfrak{g}\mid a_r(x)\neq 0\} \text{ – connected, dense, open in } \mathfrak{g}.}$$

It is clear that it is open (as the preimage of the open set $\mathbb{C}\setminus\{0\}\subseteq\mathbb{C}$ under $a_r:\mathfrak{g}\to\mathbb{C}$). It is also dense, since its complement $\{x\in\mathfrak{g}\mid a_r(x)=0\}$ cannot contain a ball.

Finally, to prove it is path connected, for any $x,y\in\mathfrak{g}^{sr}$, consider the polynomial

$$\boxed{q(t):=a_r(t\cdot x+(1-t)\cdot y) \text{ on complex line } t\in\mathbb{C}}$$

As $q(0)=a_r(y)$, $q(1)=a_r(x)$, it's a non-vanishing polynomial on $\mathbb{C}$, hence has only finitely many zeros. Therefore, x & y can be connected by a path inside $\{t\cdot x+(1-t)y\mid t\in\mathbb{C}\}$ consisting only of strongly regular el-s ◼

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Proposition 1: Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{h} \leq \mathfrak{g}$ – a Cartan subalgebra.

1) $\dim \mathfrak{h} = \operatorname{rank} \mathfrak{g}$

2) $\mathfrak{h}^{\text{reg}} := \mathfrak{h} \cap \mathfrak{g}^{s.r.}$ coincides with $\mathfrak{h}^{\text{reg}} = \{h \in \mathfrak{h} \mid \alpha(h) \neq 0 \ \forall \alpha \in R\}$

In particular, $\mathfrak{h}^{\text{reg}}$ is open and dense in $\mathfrak{h}$.

1) Let G be a connected Lie group with the Lie algebra $\mathfrak{g}$. We consider

$$\boxed{\mathcal{U} := \operatorname{Ad}G(V) \quad \text{where } V := \{h \in \mathfrak{h} \mid \alpha(h) \neq 0 \ \forall \alpha \in R\}}$$

First, we claim that $\mathcal{U}$ is open in $\mathfrak{g}$. To this end, we consider the map

$\varphi : G \times V \longrightarrow \mathfrak{g}$, $(g, X) \longmapsto \operatorname{Ad}(g)(X)$ whose image is precisely $\mathcal{U}$. Then, for any $X \in V$,

the differential φ_* at the point $(1, X)$ is a map $\varphi_* : \mathfrak{g} \oplus \mathfrak{h} \longrightarrow \mathfrak{g}$ given by

$$\varphi_* : (\underset{\mathfrak{g}}{y}, \underset{\mathfrak{h}}{h}) \longmapsto [y, X] + h.$$

As $\alpha(X) \neq 0 \ \forall \alpha \in R$, we have $[\mathfrak{g}_\alpha, X] = \mathfrak{g}_\alpha \ \forall \alpha \in R$. Evoking the root decomposition of $\mathfrak{g}$ w.r.t. $\mathfrak{h}$, we conclude that φ_* is surjective. Hence, by the inverse f-n thm, $\mathcal{U} = \operatorname{Im}(\varphi)$ contains an open nbhd of X in $\mathfrak{g}$, but then it also holds for any $\operatorname{Ad}(g)(X)$.

<u>Thus</u>: $\mathcal{U}$ is open!

As $\mathcal{U}$ is open and $\mathfrak{g}^{s.r.}$ is dense (Lemma 1), we have $\mathcal{U} \cap \mathfrak{g}^{s.r.} \neq \emptyset$. But for any $u = \operatorname{Ad}(g)(\underset{V}{X}) \in \mathcal{U}$, we have $n(u) = n(X) = \dim \mathfrak{h}$ (by root decomposition). Hence, for $u \in \mathcal{U} \cap \mathfrak{g}^{s.r.}$, we get $\operatorname{rank}(\mathfrak{g}) = n(u) = \dim \mathfrak{h}$.

2) Evoking the root decomposition of $\mathfrak{g}$, we see that for any $h \in \mathfrak{h}$:

$$n(h) = \dim \mathfrak{h} + \sum_{\alpha \in R : \alpha(h) = 0} \dim \mathfrak{g}_\alpha = \dim \mathfrak{h} + \#\{\alpha \in R \mid \alpha(h) = 0\}$$

hence $h \in \mathfrak{g}^{s.r.}$ iff $h \in V$. ■

<u>Notation-wise</u>: We note that we use $\mathfrak{h}^{\text{reg}}$ not $\mathfrak{h}^{s.r.}$ as all el-s in $\mathfrak{h}$ are semisimple. To this end, we note that there is a different notion of <u>regular elements</u>:

$$\boxed{X \in \mathfrak{g} \text{ is called } \underline{\text{regular}} \text{ if } \dim\Big(\underset{\text{not generalized!}}{\text{ordinary}} \text{ 0-eigenspace of } \operatorname{ad}(X) : \mathfrak{g} \to \mathfrak{g}\Big) = \operatorname{rank}(\mathfrak{g})}$$

According to Corollary 2 below:

$$\boxed{\underbrace{\mathfrak{g}^{s.r.}}_{\text{strongly regular}} = \underbrace{\mathfrak{g}^{\text{reg}}}_{\text{all regular elements}} \cap \underbrace{\mathfrak{g}^{ss}}_{\text{all semisimple el-s}}}$$

(<u>Exercise</u>: Describe regular elements for $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$

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Theorem 1: Let $\mathfrak{g}$ be a complex semisimple Lie algebra.

1) If $x \in \mathfrak{g}$ is a strongly regular semisimple element (such exist by Prop 1), then the centralizer $C(x) = \{y \in \mathfrak{g} \mid [x,y]=0\}$ is a Cartan subalgebra of $\mathfrak{g}$

2) Any Cartan subalgebra of $\mathfrak{g}$ is of this form.

▶ 1) Consider the igenspace decomposition of $\mathfrak{g}$ w.r.t. $\mathrm{ad}(x)$

$$\mathfrak{g} = \bigoplus_{\lambda \in \mathbb{C}} \mathfrak{g}_\lambda, \quad \mathfrak{g}_\lambda = \{y \in \mathfrak{g} \mid [x,y] = \lambda \cdot y\}$$

Note that $C(x) = \mathfrak{g}_0$ and it is reductive by [Lecture 14, Corollary 2a)], where $\mathbb{C}\cdot x$ is the toral subalgebra used in Lect. 14.

We claim that $\mathfrak{g}_0$ is nilpotent (together with $\mathfrak{g}_0$-reductive that implies $\mathfrak{g}_0$-abelian). By Engel's Theorem, it suffices to show that $\forall y \in \mathfrak{g}_0$ the restriction $\mathrm{ad}(y)|_{\mathfrak{g}_0}$ is nilpotent. As $\mathrm{ad}(x)|_{\mathfrak{g}/\mathfrak{g}_0}$ is invertible, so is $\mathrm{ad}(x_t)|_{\mathfrak{g}/\mathfrak{g}_0}$ with $x_t := x + ty$ for small enough t. But then $n(x_t) \leq \dim \mathfrak{g}_0 = \mathrm{rank}(\mathfrak{g})$ with equality iff $\mathrm{ad}(x_t)|_{\mathfrak{g}_0}$ - nilpotent. Thus, for small enough $t \in \mathbb{C}$, the operator $\underbrace{\mathrm{ad}(x+ty)}_{=t\cdot \mathrm{ad}(y)} : \mathfrak{g}_0 \to \mathfrak{g}_0$ is nilpotent, hence, so is $\mathrm{ad}(y): \mathfrak{g}_0 \to \mathfrak{g}_0$.

Next, let us show that all elements of $\mathfrak{g}_0$ are semisimple. Indeed, for any $y, z \in \mathfrak{g}_0$, the equality $[y,z]=0$ implies that $[y_n, z]=0$, where $y = y_s + y_n$ is the abstract Jordan decomposition and $y_s, y_n \in \mathfrak{g}_0$, $\mathrm{ad}(y_n): \mathfrak{g} \to \mathfrak{g}$ - nilpotent. Then: $\mathrm{ad}(y_n) \circ \mathrm{ad}(z): \mathfrak{g} \to \mathfrak{g}$ is nilpotent $\Rightarrow$ has trace 0 $\Rightarrow K^{\mathfrak{g}}(y_n, z) = 0 \quad \forall z \in \mathfrak{g}_0$. But $K^{\mathfrak{g}}|_{\mathfrak{g}_0 \times \mathfrak{g}_0}$ is non-degenerate (Lecture 16) $\Rightarrow y_n = 0 \Rightarrow y = y_s$ - semisimple $\forall y \in \mathfrak{g}_0$.

Finally, $\mathfrak{g}_0$ is a maximal toral subalg since any $y \in C(x)$ is in $y \in \mathfrak{g}_0$. This establishes $\mathfrak{g}_0 = C(x)$ - Cartan subalgebra of $\mathfrak{g}$.

2) Let $\mathfrak{h} \subseteq \mathfrak{g}$ be a Cartan subalgebra. By Prop 1, $\exists\, x \in \mathfrak{h}^{\mathrm{reg}}$ which is semisimple as all el-s of $\mathfrak{h}$ are semisimple. Then, $\mathfrak{h} = C(x)$ as discussed in Prop 1 ▪

Corollary 2: 1) Any $x \in \mathfrak{g}^{s.r.}$ is semisimple

2) Any $x \in \mathfrak{g}^{s.r.}$ is contained in a unique Cartan subalgebra $\mathfrak{h}_x = C(x)$.

▶ 1) As $p_x(t) = p_{x_s}(t)$, i.e. char. pol-s coincide, if $x \in \mathfrak{g}^{s.r.}$ then also $x_s \in \mathfrak{g}^{s.r.}$. But $C(x_s)$ is a Cartan subalgebra of $\mathfrak{g}$ (by Thm 1), $x \in C(x_s) \Rightarrow x$ - semisimple.

2) By part 1) and Thm 1, we know that $\mathfrak{h}_x := C(x)$ is a Cartan subalgebra of $\mathfrak{g}$. If $x \in \mathfrak{h}$ - another Cartan subalg $\Rightarrow \mathfrak{h} \subseteq \mathfrak{h}_x \Rightarrow \mathfrak{h} = \mathfrak{h}_x$ as $\dim \mathfrak{h} = \mathrm{rank}(\mathfrak{g}) = \dim \mathfrak{h}_x$ (Prop 1) ▪

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We conclude with the following beautiful result (Chevalley's Theorem)

Theorem 2: Any two Cartan subalgebras of a complex semisimple Lie algebra $\mathfrak{g}$ are conjugate. In other words, if $\mathfrak{h}_1, \mathfrak{h}_2$ — two Cartan subalgebras of $\mathfrak{g}$, and G is a connected Lie group with $\mathrm{Lie}(G)=\mathfrak{g}$, then
$$\exists g \in G : \mathrm{Ad}(g)(\mathfrak{h}_1) = \mathfrak{h}_2$$

▶ By Corollary 2, any $x \in \mathfrak{g}^{sr}$ is contained in a unique Cartan subalgebra $\mathfrak{h}_x$.
Define the following equivalence relation on $\mathfrak{g}^{sr}$:
$$\boxed{x \sim y \Longleftrightarrow \mathfrak{h}_x \text{ is conjugate to } \mathfrak{h}_y}$$

If $x, y \in \mathfrak{h}^{reg}$, then $\mathfrak{h}_x = \mathfrak{h}_y = \mathfrak{h}$, hence $x \sim y$. Then also $\mathrm{Ad}(g)(x) \sim y$ for such x, y.
Moreover, any $z \sim y$ must be of such form i.e. $z = \mathrm{Ad}(g)(x)$ with $x \in \mathfrak{h}_y$.

Then: the equivalence class of any $y \in \mathfrak{g}^{sr}$ is open, see Proposition 1.
But according to Lemma 1, the set $\underset{\text{∐ equivalence classes}}{\mathfrak{g}^{sr}}$ is connected. $\Big\} \Rightarrow$

$\Rightarrow$ there is just one equivalence class. $\Rightarrow$ any two Cartan subalg-s $\mathfrak{h}_x, \mathfrak{h}_y$ $(x, y \in \mathfrak{g}^{sr})$ are conjugate.

But according to Theorem 1, any Cartan subalg. $\mathfrak{h}$ of $\mathfrak{g}$ is of the form $\mathfrak{h} = \mathfrak{h}_x$ for some $x \in \mathfrak{g}^{sr}$