Lecture #18

- Last time:
  - finished the key properties of the root decomposition
  - introduced the notions of nullity, rank, strongly regular, and regular elements
  - proved the set $\mathfrak{g}^{sr}$ is connected, dense, open in $\mathfrak{g}$
  - finished with $\dim \mathfrak{h} = \operatorname{rank} \mathfrak{g}$ and $\mathfrak{h} \cap \mathfrak{g}^{sr} = \{x \in \mathfrak{h} \mid \alpha(h) \neq 0 \ \forall \alpha \in R\}$
- Go over the key idea of the last statement above (with $\mathfrak{sl}_n$-example in mind)
- Cover the rest of [Lecture #17, pages 4-5], i.e. Thm 1, Cor 2, Thm 2. ← TOOK almost all Lecture!
- Also cover Corollary 1 from Lecture #17 that allows to reduce to $\mathbb{R}$ v.space, which is important for our discussions this and next weeks.

Root Systems

Viewing R as a subset of $\mathfrak{h}^*_{\mathbb{R}}$, they possess a number of remarkable properties. As such we will develop an abstract theory of (reduced) root systems, and then relate it back to our study of simple Lie algebras.

Def 1: Let $E \simeq \mathbb{R}^r$ be a Eucledian vector space with a positive definite inner product. An abstract root system is a finite set $R \subseteq E \setminus \{0\}$ satisfying the following properties:

(R1) R spans E

(R2) $\forall \alpha, \beta \in R$, the number $n_{\alpha\beta} := \dfrac{2(\alpha,\beta)}{(\alpha,\alpha)}$ is an integer

(R3) $\forall \alpha, \beta \in R$, the reflection $s_\alpha(\beta) := \beta - n_{\alpha\beta}\cdot\alpha$, is also an element of R

Note that $s_\alpha(\alpha) = -\alpha$, hence $\forall \alpha \in R$ its negative $-\alpha$ is also in R!

Def 2: A root system R is reduced if for $\alpha, c\alpha \in R$, we have $c = \pm 1$.

! The number $r = \dim_{\mathbb{R}} E$ is called the rank of R.

Note that if $\beta = c\alpha$, then $n_{\alpha\beta} = 2c$, $n_{\beta\alpha} = \frac{2}{c}$, hence (R2) implies $c \in \{\pm 1, \pm 2, \pm \frac{1}{2}\}$

But Def 2 is indeed more restrictive than Def 1 due to the following example:

(Exercise: Verify that $R := \{\pm e_i, \pm 2e_i\}_{i=1}^n \cup \{\pm e_i \pm e_j\}_{1\le i<j\le n} \subseteq \mathbb{R}^n$ is a non-reduced root system ($\{e_i\}$ is the standard orthonormal basis of $\mathbb{R}^n$)