• Last time: abstract root systems.

We shall develop the theory of those in the next few classes and then apply it back to the theory of simple Lie algebras. To this end, it is important to view a root system $R$ of a simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ as a subset of $\mathbb{R}$-vector space.

**Proposition 1** (see [Lecture #17, Corollary 1])
Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{h}\subseteq\mathfrak{g}$ - Cartan subalgebra, so that
$$\mathfrak{g}\simeq\mathfrak{h}\oplus\bigoplus_{\alpha\in R}\mathfrak{g}_\alpha \text{ is a } \underline{\text{root decomposition}}.$$
a) Let $\mathfrak{h}_\mathbb{R}$ be the $\mathbb{R}$-span of $\{h_\alpha\}_{\alpha\in R}$. Then the restriction of the Killing form to $\mathfrak{h}_\mathbb{R}$ is positive definite and $\mathfrak{h}\simeq\mathfrak{h}_\mathbb{R}\oplus i\,\mathfrak{h}_\mathbb{R}$ ($\mathfrak{h}$ is a complexification of $\mathfrak{h}_\mathbb{R}$)

b) Let $\mathfrak{h}^*_\mathbb{R}$ be the $\mathbb{R}$-span of $\{\alpha\}$ in $\mathfrak{h}^*$. Then
$$\mathfrak{h}^*_\mathbb{R}=\{\lambda\in\mathfrak{h}^*\mid \lambda(h)\in\mathbb{R} \text{ for all } h\in\mathfrak{h}_\mathbb{R}\}=(\mathfrak{h}_\mathbb{R})^*, \text{ and } \mathfrak{h}^*\simeq\mathfrak{h}^*_\mathbb{R}\oplus i\,\mathfrak{h}^*_\mathbb{R}$$

▷ a) For any $\alpha,\beta\in R$, we have
$$(h_\alpha,h_\beta)=\mathrm{tr}(\mathrm{ad}(h_\alpha)\mathrm{ad}(h_\beta))=\sum_{\gamma\in R}\gamma(h_\alpha)\gamma(h_\beta)\in\mathbb{Z}$$
↑ Lecture #16

Likewise, for any $h=\sum_{\alpha\in R}c_\alpha h_\alpha\in\mathfrak{h}_\mathbb{R}$ (constants $c_\alpha\in\mathbb{R}$), we have
$$\forall\gamma\in R:\ \gamma(h)=\sum_{\alpha\in R}\underbrace{c_\alpha}_{\in\mathbb{R}}\underbrace{\gamma(h_\alpha)}_{\in\mathbb{Z}}\in\mathbb{R}\implies (h,h)=\sum_{\gamma\in R}\gamma(h)^2\geq 0 \quad\text{with } (h,h)=0\iff h=0.$$

Therefore, restriction of the Killing form to $\mathfrak{h}_\mathbb{R}$ is positive definite!

Hence, $\mathfrak{h}_\mathbb{R}\cap i\mathfrak{h}_\mathbb{R}=0$ (as restriction on $i\mathfrak{h}_\mathbb{R}$ is negative definite) $\Rightarrow \dim_\mathbb{R}\mathfrak{h}_\mathbb{R}\leq\dim_\mathbb{C}\mathfrak{h}$.

But since $\mathfrak{h}=\mathrm{span}_\mathbb{C}\{h_\alpha\}$, we also have $\dim_\mathbb{R}\mathfrak{h}_\mathbb{R}\geq\dim_\mathbb{C}\mathfrak{h}$. Thus, $\dim_\mathbb{R}\mathfrak{h}_\mathbb{R}=\dim_\mathbb{C}\mathfrak{h}$
$$\Rightarrow \mathfrak{h}\simeq\mathfrak{h}_\mathbb{R}\oplus i\mathfrak{h}_\mathbb{R}.$$

b) Follows immediately from a) ■

The above result produces an abstract root system from any semisimple complex Lie algebra $\mathfrak{g}$:

**Proposition 2**: Let $\mathfrak{g}$ be a semisimple complex Lie algebra with the root decomposition
$$\mathfrak{g}=\mathfrak{h}\oplus\bigoplus_{\alpha\in R}\mathfrak{g}_\alpha$$
Then, the set of roots $R\subseteq\mathfrak{h}^*_\mathbb{R}\setminus\{0\}$ is a reduced (abstract) root system.

Example: $\mathfrak{g}\simeq\mathfrak{sl}_n$, $\mathfrak{h}=$ diagonal matrices $\rightsquigarrow \mathfrak{h}^*_\mathbb{R}=\mathbb{R}^n/\mathbb{R}(1,\ldots,1)\simeq\{(\lambda_1,\ldots,\lambda_n)\in\mathbb{R}^n\mid\lambda_1+\ldots+\lambda_n=0\}$
"$A_{n-1}$-type" $R=\{e_i-e_j\mid 1\leq i\neq j\leq n\}$, $e_i=(0,\ldots,\underset{i^{\text{th}}\text{ spot}}{1},0,\ldots,0)$. All conditions (R1-R3) are obvious.

# Lecture # 19

We note that each root system admits a "dual" root system as discussed below.

$\forall \alpha \in R$, define the coroot $\alpha^\vee \in E^*$ via $\alpha^\vee(x) = \frac{2(\alpha, x)}{(\alpha,\alpha)}$. Then:

- $\alpha^\vee(\alpha) = 2$
- $n_{\alpha\beta} = \alpha^\vee(\beta)$
- $s_\alpha(\beta) = \beta - \alpha^\vee(\beta)\cdot\alpha$

**Example**: In the setup of Prop 2, with $E = \mathfrak{h}^*_{\mathbb{R}} \supset R$ – roots of $\mathfrak{g}$, we naturally have $\alpha^\vee = h_\alpha$ that was crucial back in Lect 16

**Exercise**: Let $R \subset E$ be a root system. Verify that $R^\vee := \{\alpha^\vee \mid \alpha \in R\} \subset E^*$ is also a root system (the inner form on $E^*$ is arising through that on $E$ via $E \xrightarrow{\sim} E^*$)

The root system $R^\vee$ is called a "dual root system" of $R$.

## • The Weyl group

**Def 1**: Let $R_1 \subset E_1$, $R_2 \subset E_2$ be root systems. An isomorphism of root systems $\varphi: R_1 \to R_2$ is a vector space isomorphism $\varphi: E_1 \xrightarrow{\sim} E_2$ such that

$$\boxed{\varphi(R_1) = R_2 \quad \text{and} \quad n_{\alpha\beta} = n_{\varphi(\alpha)\varphi(\beta)} \quad \forall \alpha, \beta \in R_1}$$

Note: $\varphi$ does not need to preserve inner product (e.g. can rescale it by $c \in \mathbb{R}\setminus\{0\}$)

**Def 2**: The Weyl group $W$ of a root system $R$ is the subgroup of $GL(E)$ generated by the reflections $\{s_\alpha\}_{\alpha\in R}$.

**Lemma 1**: a) The Weyl group $W$ is a finite subgroup in the orthogonal group $O(E)$, and $R$ is $W$-invariant.

b) $\forall w \in W$, $\alpha \in R$: $s_{w(\alpha)} = w s_\alpha w^{-1}$

► a) As every reflection $s_\alpha$ is an orthogonal transformation of $E$, and $s_\alpha(R) = R$, we have $W < O(E)$ and $w(R) = R$ $\forall w \in W$. As $R$ spans $E$, if $w(\alpha) = \alpha$ $\forall \alpha \in R$ then $w = id_E : E \to E$. Hence, $W$ is a finite subgroup of $O(E)$.

b) Obvious geometric statement ■

**Example** ($A_{n-1}$-type root system): $W = S_n$ – the symmetric group with $s_{e_i - e_j} \in W$ corresponding to the transposition $(ij) \in S_n$.

Note: Not all automorphisms of a root system are given by elements in $W$.

Lecture #19

Our ultimate goal is to classify all reduced (abstract) root systems, which then will be used to classify all semisimple Lie algebras. To this end, we start by classifying all root systems of rank 2.

- RANK 2

NOTE: If $\alpha, \beta$ are two linearly independent roots (i.e. $\beta \neq \pm\alpha$) in a (reduced) root system $R \subset E$, then $R' := R \cap \mathrm{Span}_{\mathbb{R}}(\alpha, \beta)$ is clearly a root system in $E' = \mathrm{Span}_{\mathbb{R}}(\alpha, \beta)$. For this reason, we shall later use rank 2 root systems to study all $R$.

**Proposition 3**: Let $R$ be a reduced root system, $\alpha, \beta \in R$ – not multiples of each other, and assume $|\alpha| \geq |\beta|$. Let $\phi$ be the angle between $\alpha$ and $\beta$.
Then, we have one of the following possibilities:

1) $\phi = \frac{\pi}{2}$, $n_{\alpha\beta} = n_{\beta\alpha} = 0$

2a) $\phi = \frac{2\pi}{3}$, $|\alpha| = |\beta|$, $n_{\alpha\beta} = n_{\beta\alpha} = -1$

2b) $\phi = \frac{\pi}{3}$, $|\alpha| = |\beta|$, $n_{\alpha\beta} = n_{\beta\alpha} = 1$

3a) $\phi = \frac{3\pi}{4}$, $|\alpha| = \sqrt{2}|\beta|$, $n_{\alpha\beta} = -1$, $n_{\beta\alpha} = -2$

3b) $\phi = \frac{\pi}{4}$, $|\alpha| = \sqrt{2}|\beta|$, $n_{\alpha\beta} = 1$, $n_{\beta\alpha} = 2$

4a) $\phi = \frac{5\pi}{6}$, $|\alpha| = \sqrt{3}|\beta|$, $n_{\alpha\beta} = -1$, $n_{\beta\alpha} = -3$

4b) $\phi = \frac{\pi}{6}$, $|\alpha| = \sqrt{3}|\beta|$, $n_{\alpha\beta} = 1$, $n_{\beta\alpha} = 3$

▶ Evoking $(\alpha, \beta) = |\alpha| \cdot |\beta| \cdot \cos\phi \Rightarrow \begin{cases} n_{\alpha\beta} = \frac{2|\beta|}{|\alpha|}\cos\phi \\ n_{\beta\alpha} = \frac{2|\alpha|}{|\beta|}\cos\phi \end{cases} \Rightarrow \underbrace{4\cos^2\phi}_{\in [0,4)} = n_{\alpha\beta} \cdot n_{\beta\alpha} \in \mathbb{Z}$ ← excluding 4 as $\alpha, \beta$ – lin. indep.

Therefore, $n_{\alpha\beta} \cdot n_{\beta\alpha} \in \{0, 1, 2, 3\}$ and both $n_{\alpha\beta}, n_{\beta\alpha}$ are integers.
Moreover, if $\cos\phi \neq 0$, then $\frac{|\alpha|^2}{|\beta|^2} = \frac{n_{\beta\alpha}}{n_{\alpha\beta}}$.

Now all the above cases follow immediately by analyzing all possibilities for $n_{\alpha\beta} \cdot n_{\beta\alpha}$. ∎

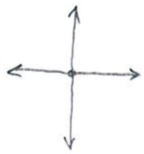
In fact, all these possibilities from Prop 3 are realized.

**Theorem 1**: a) Let $A_1 \cup A_1, A_2, B_2, G_2$ be the sets of vectors in $\mathbb{R}^2$ as depicted next. Then each of them is a rank 2 root system.

b) Any rank 2 root system is isomorphic to one of root systems from a)

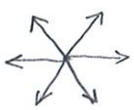
# Lecture # 19

- $A_1 \cup A_1$ root system



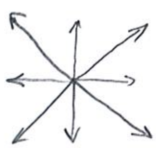
Note: the ratio of lengths of vertical to horizontal vectors is arbitrary

- $A_2$ root system (root system of $\mathfrak{g} = \mathfrak{sl}_3$)



← all angles $= \pi/3$
all vectors of the same length

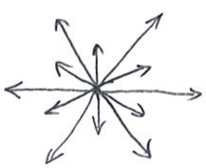
- $B_2 \simeq C_2$ root system (root system of $\mathfrak{g} = \mathfrak{so}_5$ and $\mathfrak{g} = \mathfrak{sp}_4$)



← all angles $= \pi/4$
ratios of lengths of vectors are $1, \sqrt{2}^{\pm 1}$

(vertices and midpoints of edges of a square)

- $G_2$ root system



← all angles $= \pi/6$
ratios of longer to shorter vectors are $\sqrt{3}$

(vertices of 2 regular concentric 6-gons)

## Proof of Theorem 1

a) direct verification

b) Pick lin. independent roots $\alpha, \beta \in R$ s.t. the angle $\phi$ between them is max. possible. Clearly $\phi \geq \frac{\pi}{2}$ (if not, replace $\beta$ by $-\beta$). We can also assume $|\alpha| \geq |\beta|$.
Hence, we are in one of the cases 1), 2a), 3a), 4a) from Prop 1.
As $R$ is $W$-stable (Lemma 1), we recover all the depicted vectors in each of the 4 cases by applying compositions of $s_\alpha, s_\beta$. Thus, in cases 1), 2a), 3a), 4a) our root system contains $A_1 \cup A_1$, $A_2$, $B_2$, or $G_2$, respectively.
In fact, they are equal, since otherwise one would find two vectors with a larger angle than $\phi$ (case-by-case consideration)

Lecture #19

As an immediate useful corollary of Theorem 1, we obtain:

**Corollary 1**: If two roots $\alpha, \beta$ in (abstract) root system $R$ are linearly independent and $(\alpha,\beta)<0$, then $\alpha+\beta \in R$.

▷ Using the observation on top of p. 3, it suffices to check for rank 2 root systems. But according to Theorem 1b), there are just 4 cases to check, where it is straightforward ∎

• Positive and simple roots

To proceed with the classification of reduced root systems, it's desirable to find the smallest set of "generating roots", similar to our treatment of rank 2.

Fix a reduced root system $R \subset E$. Pick $t \in E^*$ s.t. $t(\alpha) \neq 0 \ \forall \alpha \in R$

[ Note: Identifying $E \simeq E^*$ through inner product on $E$, we could alternatively look at $t \in E$ s.t. $(t,\alpha) \neq 0 \ \forall \alpha \in R$

**Def 3**: a) $\alpha \in R$ is *positive* (with respect to $t$) if $t(\alpha)>0$. Let $R_+ = \{\text{all positive roots}\}$

b) $\alpha \in R$ is *negative* (w.r.t. $t$) if $t(\alpha)<0$. Let $R_- = \{\text{all negative roots}\} = -R_+$

c) the decomposition $R = R_+ \cup R_-$ is called a *polarization* of $R$

Example ($A_{n-1}$-root system): $R = \{e_i - e_j \mid 1 \le i \neq j \le n\}$, hence, $t(\alpha) \neq 0 \ \forall \alpha \in R$ if $t = (t_1, \ldots, t_n)$ with $t_i \neq t_j \ \forall i \neq j$. For specific choice $t_1 > t_2 > \ldots > t_n$, we see that $R_+ = \{e_i - e_j \mid i<j\}$. Moreover, it's clear in this case that polarizations of $R$ are in natural bijection with $S_n$ = Weyl group (we shall generalize this later to all root systems)

**Def 4**: A root $\alpha \in R_+$ is *simple* if it is not a sum of two positive roots i.e. $\nexists \beta, \gamma \in R_+$ s.t. $\alpha = \beta + \gamma$.

The set of simple roots will be denoted by $\Pi \subseteq R_+$

The following is obvious (induction on $t(\alpha)$):

**Lemma 2**: Every positive root $\alpha \in R_+$ can be written as a sum of simple roots

Lecture #19

We also have:

**Lemma 3:** If $\alpha, \beta \in \Pi \subset R_+$, $\alpha \neq \beta$, then $(\alpha, \beta) \le 0$

▶ If $(\alpha,\beta)>0$, then $(-\alpha,\beta)<0 \overset{\text{Cor 1}}{\Longrightarrow} \underbrace{\beta-\alpha}_{=:\gamma} \in R = R_+ \cup R_-$

- if $\beta - \alpha \in R_+ \Rightarrow \beta = \alpha + \gamma$, contradiction with $\beta \in \Pi$
- if $\beta - \alpha \in R_- \Rightarrow \alpha = \beta + (-\gamma)$, contradiction with $\alpha \in \Pi$

So: $(\alpha, \beta) \le 0$ ■

**Theorem 2:** The set $\Pi \subset R_+$ of simple roots is a basis of $E$

( Exercise: Prove this theorem using Lemma 3

**Corollary 2:** Every $\alpha \in R$ can be uniquely written as a linear combination of simple roots with integer coefficients:

$$\alpha = \sum_{i=1}^{r} n_i \alpha_i, \quad n_i \in \mathbb{Z}, \quad \Pi = \{\alpha_1, \alpha_2, \dots \alpha_r\}$$

Moreover, all $n_i \ge 0$ for $\alpha \in R_+$, and all $n_i \le 0$ for $\alpha \in R_-$

**Def 5:** For a positive root $\alpha \in R_+$, its *height* is $ht(\alpha) = \sum n_i \in \mathbb{Z}_{>0}$
↑ same as in Cor 2

This is a useful notion, as often results can be proved by induction on $ht$.

**Example** ($A_{n-1}$ root system): For $R = \{e_i - e_j \mid 1 \le i \ne j \le n\}$ and $R_+ = \{e_i - e_j \mid i<j\}$, the simple roots are $\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \dots, \alpha_{n-1} = e_{n-1} - e_n$, so that $e_i - e_j = \alpha_i + \alpha_{i+1} + \dots + \alpha_{j-1} \;\; \forall i<j \Rightarrow ht(e_i - e_j) = j-i$.

**Exercise:** a) For $\mathfrak{g} = sp_{2n}(\mathbb{C})$, recall the root system $R$ from [Hwk8, Prob 4(b)]. It's called *$C_n$-type* root system. Compute simple roots for the polarization $R_+ = \{e_i - e_j \mid 1 \le i < j \le n\} \cup \{e_i + e_j \mid 1 \le i \le j \le n\}$

b) For $\mathfrak{g} = so_{2n}(\mathbb{C})$, recall the *$D_n$-type* root system $R$ from [Hwk8, Pr. 4(c)]. Compute simple roots for polarization $R_+ = \{e_i \pm e_j \mid 1 \le i < j \le n\}$

c) For $\mathfrak{g} = so_{2n+1}(\mathbb{C})$, recall the so-called $B_n$-type root system $R$ from [Hwk8, Pr. 4(d)]. Compute $\Pi$ for polarization $R_+ = \{e_i \pm e_j \mid i<j\} \cup \{e_i\}_{i=1}^{n}$.