

Last time

- root systems $R \subseteq \mathfrak{h}_{\mathbb{R}}^*$ arising from semisimple Lie algebras

- Weyl group $W \subset O(E)$ of any root system $R \subset E$

- complete classification of rank 2 root systems

↳ Corollary: $\alpha, \beta \in R$ - linearly indep. roots with $(\alpha, \beta) < 0 \Rightarrow \alpha + \beta \in R$.

- polarization of $R = R_+ \cup R_-$ AND simple roots $\Pi \subseteq R_+$

↳ Work out example of A_n -type root system

- we ended last time with the following key result left as a Hwk Exercise:

Thm: $\Pi \subset R_+$ is a basis of E

↳ Discuss [Lecture 19, Lemma 3 & Corollary 2 & Def 5] today.

• Discuss Root systems of types B_n, C_n, D_n , their relation to orthogonal/symplectic Lie algebras in the context of [Hwk 8, Problem 4] (may postpone till Dyubin diagrams's discussion) [Hwk 9, Problem 4].

Root and Weight lattices

A lattice in a real vector space E is a subgroup $L \subset E$ generated by a basis of E (hence, by a suitable change of basis, any lattice can be identified with $\mathbb{Z}^n \subseteq \mathbb{R}^n$). Moreover, given a lattice $L \subset E$ one has a natural dual lattice

$$E^* \supset L^* := \{f \in E^* \mid f(x) \in \mathbb{Z} \forall x \in L\}$$

[Clear: If L is generated (over \mathbb{Z}) by a basis $\{e_i\}_{i=1}^n$, then L^* is generated by the dual basis $\{e_i^*\}_{i=1}^n$ defined via $e_i^*(e_j) = \delta_{ij} = \begin{cases} 1, & \text{if } i=j \\ 0, & \text{if } i \neq j \end{cases}$.

In the context of abstract root systems $R \subset E$, there are several lattices that are very important:

1) root lattice $Q \subset E$ is the lattice generated (over \mathbb{Z}) by R

If we pick a polarization $R = R_+ \cup R_-$ and $\Pi \subset R_+$ is the set of simple roots, then

$$Q = \bigoplus_{\alpha_i \in \Pi} \mathbb{Z} \alpha_i.$$

2) coroot lattice $Q^\vee \subset E^*$ generated by $\{\alpha^\vee \mid \alpha \in R\}$, which is just the root lattice of the dual root system $R^\vee \subset E^*$, see [Huk 9, Prob 2]

$$Q^\vee = \bigoplus_{\alpha \in \Pi} \mathbb{Z} \alpha^\vee$$

3) weight lattice $P \subset E$ is defined as the dual of Q^\vee :

$$P = (Q^\vee)^* = \{\lambda \in E \mid \alpha^\vee(\lambda) \in \mathbb{Z} \quad \forall \alpha \in R\}$$

4) coweight lattice $P^\vee \subset E^*$ is defined as the dual of Q or equivalently as a weight lattice of the dual root system

$$P^\vee = Q^* = \{\lambda \in E^* \mid \lambda(\alpha) \in \mathbb{Z} \quad \forall \alpha \in R\}$$

Note: $\forall \alpha, \beta \in R: \alpha^\vee(\beta) = \langle \alpha, \beta \rangle \in \mathbb{Z} \Rightarrow Q \subset P \text{ and } Q^\vee \subset P^\vee$

If we fix a polarization $R = R_+ \cup R_-$ and denote simple roots by $\Pi = \{\alpha_i\}_{i=1}^r$,

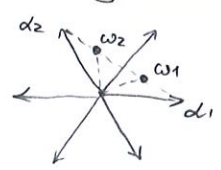
then $P = \bigoplus_{i=1}^r \mathbb{Z} \omega_i$, where the fundamental weights ω_i are s.t. $\omega_i = (\alpha_i^\vee)^*$
 $P^\vee = \bigoplus_{i=1}^r \mathbb{Z} \omega_i^\vee$, where the fundamental coweights ω_i^\vee are s.t. $\omega_i^\vee = (\alpha_i)^\vee$,

i.e. $\alpha_j^\vee(\omega_i) = \omega_i^\vee(\alpha_j) = \delta_{ij}$.

Example: a) A_1 -type root system (d -simple root)

$R = \{\pm \alpha\} \subset E \leftarrow \text{dim } \mathbb{R}\text{-v. space}$ with basis $\{\alpha\} \Rightarrow \begin{cases} Q = \mathbb{Z} \cdot \alpha \\ P = \mathbb{Z} \cdot \omega = \frac{1}{2} \mathbb{Z} \cdot \alpha \end{cases}$
 Recall that $\alpha^\vee(\alpha) = 2 \Rightarrow \omega = \frac{1}{2} \alpha$

b) A_2 -type root system



$$Q = \mathbb{Z} \alpha_1 \oplus \mathbb{Z} \alpha_2$$

$$P = \mathbb{Z} \omega_1 \oplus \mathbb{Z} \omega_2$$

| Discuss the precise location of ω_1, ω_2 in class

Exercise: a) Show that $|P/Q| = \det(A)$, $A = (\alpha_i^\vee(\alpha_j))$ - the Cartan matrix

b) Compute $|P/Q|$ for root systems of types A_n, B_n, C_n, D_n .

c) Identify explicitly the finite group P/Q for types A_n, B_n, C_n, D_n .

Lecture #20

Weyl chambers

Last time we saw that each $t \in E^*$ st. $t(\alpha) \neq 0 \forall \alpha \in R$ gives rise to a polarization $R = R_+ \cup R_-$ and consequently to the set of simple roots $\Pi \subset R_+$.

There are two natural questions:

- Q1: Do different polarizations give rise to equivalent sets of simple roots?
- Q2: Can we recover all root system R just from the set Π of simple roots?

Note that our definition of a polarization depends on a choice of

$$t \in E \setminus \bigcup_{\alpha \in R} L_\alpha, \text{ where } L_\alpha = \{ \lambda \in E \mid (\lambda, \alpha) = 0 \}$$

[Here, we already identified $E \cong E^*$ through the inner product on E]

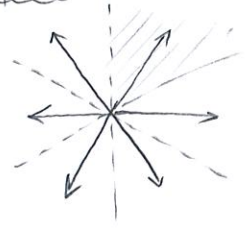
Clearly, the polarization of R does not change when t continuously deforms without crossing any of the hyperplanes L_α . With this in mind, we introduce:

Def 1: A Weyl chamber is a connected component of $E \setminus \bigcup_{\alpha \in R} L_\alpha$

Therefore, a Weyl chamber is defined by a system of strict inequalities $\pm(\alpha, \lambda) > 0$ with $\alpha \in R$.

Any such system of inequalities determines either an empty set or a Weyl chamber.

Example: A_2 -root system



dotted lines are our L_α 's
describe chambers in class!

The following result is geometrically obvious:

- Lemma 1: a) The closure \bar{C} of a Weyl chamber C is a convex cone
- b) The boundary of \bar{C} is a union of codimension 1 faces F_i , which are convex cones inside one of the root hyperplanes L_α

The root hyperplanes L_α containing the faces F_i are called the walls of C .

Lecture #20

Having fixed a polarization $R = R_+ \cup R_-$ of the root system R determines the correspondingly positive Weyl chamber C_+

$$C_+ = \{ \lambda \in E \mid (\lambda, \alpha) > 0 \ \forall \alpha \in R_+ \} = \{ \lambda \in E \mid (\lambda, \alpha_i) > 0 \ \forall \alpha_i \in \Pi \}$$

$$= \{ \sum_{\alpha_i \in \Pi} c_i \omega_i \mid c_i \in \mathbb{R}_{>0} \}$$

where ω_i are the fundamental weights

Clearly, C_+ has $r = rk(R)$ faces $L_{\alpha_1} \cap \bar{C}_+, \dots, L_{\alpha_r} \cap \bar{C}_+ \Rightarrow$ walls are $\{L_{\alpha_i}\}_{i=1}^r$.

Lemma 2: This defines a bijection between all polarizations of R and the set of Weyl chambers

(Exercise: Prove Lemma 2)

Let's now recall the Weyl group W . We know $w(\alpha) \in R \ \forall \alpha \in R, w \in W$. Hence, W -action on E maps root hyperplanes to root hyperplanes, giving rise an action of W on the set of Weyl chambers.

Theorem 1: The Weyl group W acts transitively on the set of Weyl chambers

The proof is based on the notion of adjacent chambers:

Def 2: Two Weyl chambers C, C' are adjacent if they have a common codimensional 1 face F .

The proof of theorem is based on the following result (geometrically clear):

Lemma 3: a) If C, C' are adjacent Weyl chambers separated by a hyperplane L_{α} , then $s_{\alpha}(C) = C'$

b) Any two chambers C, C' can be connected through a sequence of adjacent chambers, i.e. $C_0 = C, C_1, \dots, C_k = C'$ s.t. C_i is adjacent to C_{i+1}

Lemma 3 \Rightarrow Theorem 1 is obvious now.

Corollary 1: Every Weyl chamber has $r = rk(R)$ walls

For the positive Weyl chamber C_+ , we saw that walls are $\{L_{\alpha_i} \mid \alpha_i \in \Pi\}$ and there are exactly r of those. As every Weyl chamber C can be written as $C = w(C_+)$ for some $w \in W$ by Thm 1, all of them have the same number of walls □

Corollary 2: Any two polarizations of R are related by the action of an element $w \in W$. Thus, if Π and Π' are systems of simple roots corresponding to two polarizations $R = R_+ \cup R_-$ and $R = R'_+ \cup R'_-$, then $\exists w \in W$ s.t. $w(\Pi) = \Pi'$

Due to the bijection

$$\{\text{polarizations of } R\} \xleftrightarrow{1:1} \{\text{Weyl chambers}\}$$

from Lemma 2, and the fact ^(Theorem 1) that W acts transitively on the set of Weyl chambers, we get $W \curvearrowright \{\text{polarizations of } R\}$ transitively.

It is clear that if $w(R_+) = R'_+$, then $w(\Pi) = \Pi'$ □