Lecture #22

- Last time:
  - fixing a polarization, any chamber C can be obtained from the positive Weyl chamber C_+ through a sequence of simple reflections
  - Weyl group W is generated by simple reflections
  - all roots are recovered from the simple ones via $R = W(\Pi)$
  - Length function $\ell: W \to \mathbb{Z}_{\geq 0}$ $\rightsquigarrow$ # root hyperplanes separating C_+ & $w(C_+)$
    $\rightsquigarrow$ $\#\{\alpha \in R_+ \mid w(\alpha) \in R_-\}$
    $\rightsquigarrow$ shortest decomposition $w = s_{i_1} \dots s_{i_\ell}$
  - reduced decompositions of $w \in W$
  - $W \curvearrowright \{\text{Weyl chambers}\}$ simply transitively
  - the longest element w_0 in the Weyl group.

Exercise (Hwk 10): a) If $w = s_{i_1} \dots s_{i_\ell}$ is a reduced decomposition of $w \in W$, then one can explicitly list all ℓ roots $\alpha \in R_+$ s.t. $w(\alpha) \in R_-$:
$\{\alpha \in R_+ \mid w(\alpha) \in R_-\} = \{\beta_1, \beta_2, \dots, \beta_\ell\}$ with $\beta_k = s_{i_\ell} \dots s_{i_{k+1}}(\alpha_{i_k})$

b) Applying above to $w_0 \in W$ - the longest elt, we thus obtain an ordering on the set R_+. Its important property is that if $\alpha, \beta, \alpha+\beta \in R_+$, then:

$$\alpha < \alpha+\beta < \beta \qquad \text{or} \qquad \beta < \alpha+\beta < \alpha$$

c**) Any such order on R_+ arises from a reduced decomposition of w_0.

Lecture #22

• Cartan matrices and Dynkin diagrams

Goal: Classify all reduced root systems, and use it to classify all semisimple $\mathfrak{g}$.
As R is determined by the set Π of simple roots (Lecture #21 Theorem 1), we need to classify these.

But first, we shall reduce the problem to irreducible root systems. To this end, we note that if $R_1 \subset E_1$ and $R_2 \subset E_2$ are two root systems, then $R = R_1 \cup R_2 \subset E = E_1 \oplus E_2$ is a root system (where $R_1 \perp R_2$). Moreover, if $t_1 \in E_1$, $t_2 \in E_2$ define polarizations of R_1, R_2, then $t = t_1 + t_2 \in E$ defines a polarization of R with $\boxed{\Pi = \Pi_1 \cup \Pi_2 \;,\; \Pi_1 \perp \Pi_2}$ where Π_i = simple roots of R_i.

Def 1: A root system R is irreducible if it cannot be written as
$$R = R_1 \cup R_2, \quad R_1 \perp R_2, \quad R_{1,2} \neq \emptyset$$

Lemma 1: If R is a root system with simple roots $\Pi = \Pi_1 \cup \Pi_2$, $\Pi_1 \perp \Pi_2$, then $R = R_1 \cup R_2$ with R_i being the root system generated by Π_i

►

$\forall \alpha \in \Pi_1, \beta \in \Pi_2 : (\alpha, \beta) = 0 \Rightarrow s_\alpha(\beta) = \beta,\; s_\beta(\alpha) = \alpha$

In particular: $s_\alpha s_\beta(\gamma) = s_\alpha(\gamma - \beta^\vee(\gamma)\beta) = s_\alpha(\gamma) - \beta^\vee(\gamma)\beta = \gamma - \alpha^\vee(\gamma)\alpha - \beta^\vee(\gamma)\beta$

$\forall \gamma: \; s_\beta s_\alpha(\gamma) = \ldots$

$\Rightarrow s_\alpha$ & s_β commute. Let W_i be the subgp generated by $\{s_\alpha\}_{\alpha \in \Pi_i}$, $i = 1, 2$

Then: $W = W_1 \times W_2$, and $\left.\begin{array}{l} W_2 \text{ acts trivially on } \Pi_1 \\ W_1 \text{ acts trivially on } \Pi_2 \end{array}\right\} \Rightarrow$

$\Rightarrow R = W(\Pi) = (W_1 \times W_2)(\Pi_1 \cup \Pi_2) = W_1(\Pi_1) \cup W_2(\Pi_2) = R_1 \cup R_2$ ■

Considering the maximal decomposition of Π into mutually orthogonal subsets, we get:

Corollary 1: Any root system is uniquely a union of irreducible mutually orthogonal root systems

Thus, it suffices to classify all irreducible reduced root systems. We shall encode those by using Cartan matrices:

Def 2: The Cartan matrix of simple roots $\Pi \subset R_+$ is the matrix $(a_{ij})_{i,j=1}^r = A$ with
$$\boxed{a_{ij} = \alpha_i^\vee(\alpha_j) = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}}$$

Lecture #22

The following properties of the Cartan matrix $(\alpha_i^\vee(\alpha_j)) =: A$ are obvious:

Lemma 2: a) $a_{ii} = 2$

b) $\forall i \neq j$: $a_{ij} \in \mathbb{Z}_{\leq 0}$ AND $a_{ij} = 0 \Longleftrightarrow a_{ji} = 0$

c) $\forall i \neq j$: $a_{ij} a_{ji} = 4\cos^2\phi \in \{0, 1, 2, 3\}$, where ϕ is the angle b/w α_i & α_j

If $\phi \neq \frac{\pi}{2}$, then $\frac{|\alpha_i|^2}{|\alpha_j|^2} = \frac{a_{ji}}{a_{ij}}$

d) Let $d_i := |\alpha_i|^2$. Then the matrix $(d_i a_{ij}) = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_r \end{pmatrix} \cdot A$ is symmetric and positive definite.

It is convenient to encode Cartan matrices in the following graphical way:

Def 3: Let $\Pi = \{\alpha_1, \ldots, \alpha_r\}$ be a set of simple roots of a root system R.

The Dynkin diagram of Π is the graph constructed as follows:

* indices $i = 1, \ldots, r$ parametrize vertices of the graph
* vertices i, j are connected by $a_{ij} \cdot a_{ji}$ edges
* if $a_{ij} \neq a_{ji}$ (i.e. $|\alpha_i|^2 \neq |\alpha_j|^2$) then the arrow on the lines goes towards the shorter root.

The following result is simple:

Lemma 3: Let Π be a set of simple roots of a reduced root system R.

a) The root system R is irreducible iff the Dynkin diagram of Π is connected

b) The Dynkin diagram determines the Cartan matrix

c) R is determined by the Dynkin diagram uniquely, up to isomorphism.

▶ a) Clear, see Lemma 1.

b) $\forall i \neq j$, the edges b/w vertices i, j uniquely determine the corresponding numbers $a_{ij} = n_{\alpha_i, \alpha_j}$ and $a_{ji} = n_{\alpha_j, \alpha_i}$.

c) Follows from b) and [Thm 1(b), Lecture #21]. ■

Exercise: Show that any isomorphism b/w two irreducible root systems is a composition of a scalar operator and an isometry.

Thus, it suffices to classify all connected Dynkin diagrams.

Note: Only property d) of Lemma 2 is not clearly visible from the Dynkin diagram!

Lecture #22

Main Theorem (classification of Dynkin diagrams):

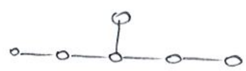
a) Connected Dynkin diagrams are classified by the following list:

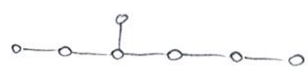
A_n $(n \geq 1)$ 

B_n $(n \geq 2)$  # vertices $= n$

C_n $(n \geq 3)$ 

D_n $(n \geq 4)$ 

E_6 

E_7 

E_8 

F_4 

G_2 

b) Every matrix satisfying the conditions of Lemma 2 is a Cartan matrix of some root system.

This result provides a full classification of irreducible reduced root systems.

Rmk:
- $A_1 = B_1 = C_1$
- $B_2 = C_2$
- $A_3 = D_3$
- $A_1 \cup A_1 = D_2$

this explains the range of parameter n above

Exercise: a) Using [Homework 9, Problem 4], verify that root systems of types A_n, B_n, C_n, D_n have Dynkin diagrams as depicted above. Write down the corresponding Cartan matrices.

b) Using [Homework 9, Problem 5], verify that root systems of types E_6, E_7, E_8, F_4, G_2 have Dynkin diagrams as depicted above. Write down the corresponding Cartan matrices.

Exercise: Recall $\rho \in E$ from [Lecture 21, Lemma 2] given by $\alpha_i^\vee(\rho)=1$ $\forall i$.
Let $\rho^\vee \in E^*$ be the dual notion (for dual root system), i.e. $\rho^\vee(\alpha_i)=1$ $\forall i$.
a) Compute $\rho, \rho^\vee$ for root systems A_n, B_n, C_n, D_n
b*) Compute $\rho, \rho^\vee$ for exceptional root systems E_6, E_7, E_8, F_4, G_2.

Def 4: A Dynkin diagram is called *simply laced* (same terminology for root systems) if all edges are simple, i.e. $a_{ij} \in \{0,-1\}$ $\forall i \neq j$.

This is equivalent to all roots having the same length.
Looking at the list from the Main Thm, we see that connected simply laced Dynkin diagrams are:

$$\underbrace{A_n (n\geq 1), \quad D_n (n \geq 4), \quad E_6, E_7, E_8}_{\text{"ADE types"}}$$

The other connected Dynkin diagrams are not simply laced, but have roots of only two possible lengths: the ratio of squared lengths is 2 for types B_n, C_n, F_4 and 3 for type G_2.

Def 5: For non-simply laced root systems, the roots of the bigger length are called *long roots*, and the rest are called *short roots*.

Exercise: a) If two vertices in a Dynkin diagram are connected by a single edge, then the corresponding simple roots are in the same W-orbit
b) Prove that for an irreducible reduced root system, the Weyl gp acts transitively on the set of all roots of the same length

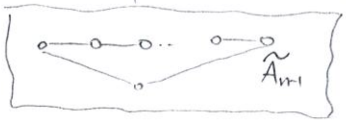
Lecture #22

Proof of the Main Theorem

▶ It remains to show that there are no other connected Dynkin diagrams.

Key Idea: As every subgraph of a Dynkin diagram is again a Dynkin diagram (specifically, property d) of Lemma 2 is preserved), we will exclude certain graphs as possible subgraphs (the corresponding matrices are degenerate). [In fact, these subgraphs will arise exactly as so-called affine Dynkin diagrams]

∘ Step 1: A Dynkin diagram cannot have a cycle (with simple or multiple edges)

Indeed, if there is a cycle with simple edges, i.e. 

then its Cartan matrix $\begin{pmatrix} 2 & -1 & & & -1 \\ -1 & 2 & -1 & 0 & \\ & -1 & 2 & \ddots & \\ & 0 & & \ddots & -1 \\ -1 & & & -1 & 2 \end{pmatrix}$ is degenerate

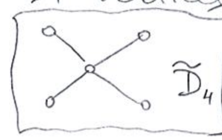
(explicitly, a vector $\boxed{(1,1,\ldots,1)=:\alpha \text{ has } (\alpha,\alpha)=0}$)

If there is a cycle with possibly multiple edges, then the corresponding Cartan matrix is $\begin{pmatrix} 2 & a_{12} & & & a_{1n} \\ a_{21} & 2 & a_{23} & 0 & \\ & a_{32} & 2 & \ddots & \\ & 0 & & \ddots & a_{n-1,n} \\ a_{n1} & & & a_{n,n-1} & 2 \end{pmatrix}$ with all $a_{ij}\le -1$ $\Rightarrow$ sum of els in each row ≤ 0

$\Rightarrow$ sum of elements in each row of symmetrized matrix is ≤ 0 $\Rightarrow$ $\boxed{(\alpha,\alpha)\le 0 \text{ for } \alpha=(1,\ldots,1)}$

$\boxed{\text{So: A Dynkin diagram is a tree.}}$

∘ Step 2: A Dynkin diagram cannot have a vertex connected to ≥ 4 vertices.

Indeed, if this happens and connecting edges are simple, we get 

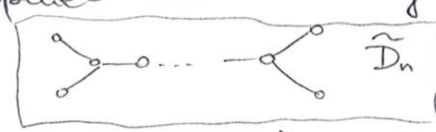
The corresponding Cartan matrix $= \begin{pmatrix} 2 & 0 & -1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 \\ -1 & -1 & 2 & -1 & -1 \\ 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 2 \end{pmatrix}$ is degenerate

(explicitly, a vector $\alpha:=(1,1,2,1,1)$ satisfies $(\alpha,\alpha)=0$)

(Exercise: Show that if some edges in $\tilde{D}_4$ are replaced by multiple, it's even worse.

$\boxed{\text{So: Every vertex of a Dynkin diagram is } \le 3\text{-valent (i.e. connected to } \le 3 \text{ vertices)}}$

∘ Step 3: There is at most one 3-valent vertex in a Dynkin diagram.

Indeed, if the reverse happened and connecting edges were all simple, we would get a subgraph 

The corresponding Cartan matrix is degenerate (explicitly, $(\alpha,\alpha)=0$ for $\alpha=$ $\begin{smallmatrix}1 & & & & & & 1\\ & >2 & -2 & -\ldots- & 2< & \\ 1 & & & & & & 1\end{smallmatrix}$)

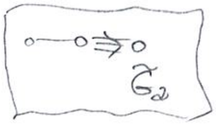
(Exercise: Show that if some edges in $\tilde{D}_n$ are replaced by multiple, it's even worse

$\boxed{\text{So: There is \underline{at most one} 3-valent vertex in a Dynkin diagram.}}$

Lecture #22
(Continuation)

- Step 4 : The only Dynkin diagram with a triple edge is G_2.

If the other edges are simple, we would otherwise get subgraphs



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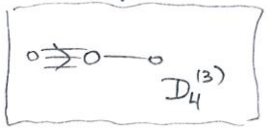
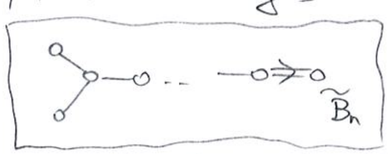


(Exercise : Show that Cartan matrices (of size 3x3) for $\tilde{G}_2$, $D_4^{(3)}$ or their versions with multiple edge instead of single one are not positive definite.

So : Unless we have G_2-type, we may assume all edges are single or double

- Step 5 : If there is a 3-valent vertex, then all edges must be simple.

If not, we would spot a subgraph



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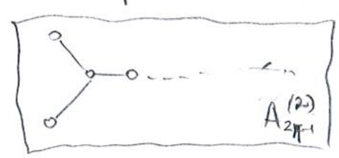
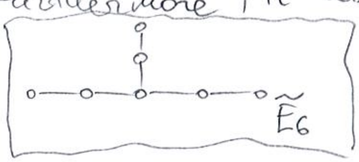


or their versions with one/two of his legs  being double.

(Exercise : Show that the corresponding Cartan matrices are not positive definite.

Furthermore, it also cannot contain any of the following:



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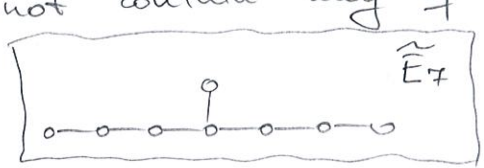

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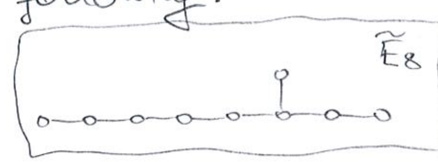
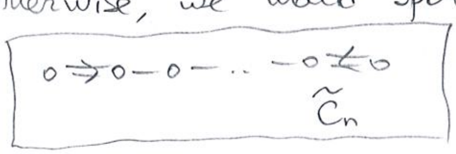


(Exercise : Show that $\tilde{E}_6, \tilde{E}_7, \tilde{E}_8$ Cartan matrices are not pos. definite.

So : If there is a 3-valent vertex in a connected Dynkin diagram, it must be one of:
$D_n (n \geq 4)$, E_6, E_7, E_8

- Step 6 : If all vertices are ≤ 2-valent, then we cannot have two double edges.

Otherwise, we would spot one of the following subgraphs:



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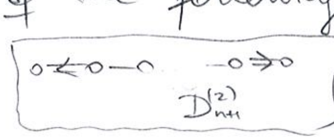

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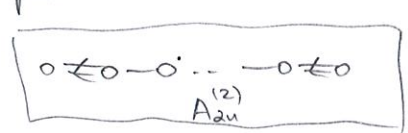


(Exercise : Show that Cartan matrices of $\tilde{C}_n$, $D_{n+1}^{(2)}$, $A_{2n}^{(2)}$ are not positive definite.

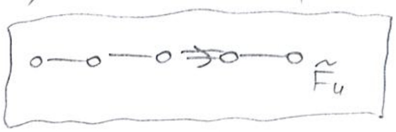
So : There are at most one double edges in a Dynkin diagram

If there are no 3-valent vertices, triple edges, double edges $\rightsquigarrow$ get A_n-type.

Lecture #22
(Continuation)

- If a double edge is at the end ⇝ get types B_n, C_n.
- Step 7: If the double edge is not in the end, then it's F_4 Dynkin diagram.

If not, we would find one of the following subgraphs:



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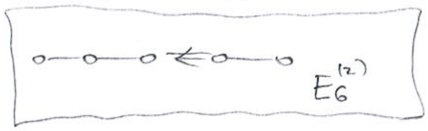

(Exercise: Show that the corresponding Cartan matrices are not positive definite.

∎