Lecture #23

Know: 1) $\mathfrak{g}$ - semisimple $\mathbb{C}$ Lie algebra $\rightsquigarrow R \subseteq \mathfrak{h}^*_{\mathbb{R}}$ is a reduced (abstract) root system

2) have a full classification of reduced root theorems - Main Theorem of Lecture 22

$\boxed{Q}$ Does every reduced root system arises through a semisimple fin.dim. $\mathbb{C}$ Lie algebra?

The goal for today is to provide the affirmative answer to this question.

We start with the following rather simple result:

Theorem 1: Let $\mathfrak{g}$ be a semisimple Lie algebra with a root system $R \subset \mathfrak{h}^*$, $(-,-)$ be a nondegenerate invariant symmetric bilinear form on $\mathfrak{g}$, $R = R_+ \cup R_-$ be a polarization, and $\Pi = \{\alpha_1, \ldots, \alpha_r\}$ - the set of simple roots. Then:

1) The subspaces $\mathfrak{n}_\pm := \bigoplus_{\alpha \in R_\pm} \mathfrak{g}_\alpha$ are subalgebras of $\mathfrak{g}$

AND $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ as a vector space

2) Choose $e_i \in \mathfrak{g}_{\alpha_i}$, $f_i \in \mathfrak{g}_{-\alpha_i}$ so that $(e_i, f_i) = \frac{2}{(\alpha_i, \alpha_i)}$, and define $h_i = h_{\alpha_i} \in \mathfrak{h}$ as in Lecture 16, so that $\{e_i, f_i, h_i\}$ span an sl_2-subalgebra $sl(2,\mathbb{C})_{\alpha_i}$.

Then $\mathfrak{n}_+$ is generated by $\{e_i\}_{i=1}^r$, $\mathfrak{n}_-$ - by $\{f_i\}_{i=1}^r$, $\mathfrak{h}$ has a basis $\{h_i\}_{i=1}^r$.

Therefore, $\mathfrak{g}$ is generated by $\{e_i, f_i, h_i\}_{i=1}^r$, as a Lie algebra.

3) The following relations hold:

(R1) $[h_i, h_j] = 0$ $\quad \forall i,j$

(R2) $[h_i, e_j] = a_{ij} e_j$, $[h_i, f_j] = -a_{ij} f_j$ $\quad \forall i,j$

(R3) $[e_i, f_j] = \delta_{ij} h_i$ $\quad \forall i,j$

(R4) $\mathrm{ad}(e_i)^{1-a_{ij}}(e_j) = 0 \iff \underbrace{[e_i, [e_i, \ldots [e_i}_{1-a_{ij}}, e_j]\ldots]] = 0$ $\quad \forall i \neq j$

(R5) $\mathrm{ad}(f_i)^{1-a_{ij}}(f_j) = 0 \iff [f_i, [f_i, \ldots [f_i, f_j]\ldots]] = 0$ $\quad \forall i \neq j$

[Exercise: Verify that $\mathfrak{n}_\pm$ are nilpotent and $\mathfrak{b}_\pm := \mathfrak{h} \oplus \mathfrak{n}_\pm$ are solvable.

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Proof of Theorem 1

▸ 1) Follows from $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$, since $\forall \alpha, \beta \in R_+$ either $\alpha+\beta$ is not a root or is a positive root.

The vector space decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ is obvious.

2) The fact that $\{h_i\}_{i=1}^r$ form a basis of $\mathfrak{h}$ follows from the general statement that $\Pi = \{\text{simple roots}\}$ form a basis of the underlying v.space

Let us now verify that $\mathfrak{n}_+$ is generated by $\{e_i\}_{i=1}^r$. Pick any $\alpha \in R_+ \setminus \Pi$.

$\boxed{\text{Claim 1: } \exists i \text{ s.t. } (\alpha, \alpha_i) > 0}$

▹ $\alpha \in R_+ \Rightarrow \alpha = c_1\alpha_1 + \ldots + c_r\alpha_r, \; c_i \in \mathbb{Z}_{\geq 0}$

$(\alpha \notin \Pi \Rightarrow$ at least two of c_i's are $>0)$

If $(\alpha, \alpha_i) \leq 0 \; \forall i \Rightarrow 0 < (\alpha, \alpha) = \sum c_i(\alpha, \alpha_i) \leq 0 \Rightarrow$ ↯ □

Thus, $(\alpha, -\alpha_i) < 0$ for some $i \in \{1, \ldots, r\} \Longrightarrow \alpha - \alpha_i \in R$ by [Lect 19, Corollary 1].

Set $\beta := \alpha - \alpha_i$, so that $\underbrace{\alpha = \beta + \alpha_i \text{ and } \beta \in R = R_+ \cup R_-. \text{ Clearly, } \beta \in R_+!}$

⇓ [Lecture 16, Thm 2]

$$\boxed{\mathfrak{g}_\alpha = [\mathfrak{g}_\beta, \mathfrak{g}_{\alpha_i}]}$$

Hence, arguing by induction on the height of roots (note: $ht(\beta) < ht(\alpha)$), we get $\mathfrak{n}_+ = \langle e_i \rangle_{i=1}^r$, i.e. $\mathfrak{n}_+$ is generated by $\{e_i\}_{i=1}^r$.

The proof for $\mathfrak{n}_-$ is analogous (or take the opposite polarization).

3) – Relations (R1), (R2) are obvious ($\mathfrak{h}$ - Cartan $\Rightarrow$ abelian; $\mathfrak{g}_\alpha$ - joint eigenspaces for $ad(\mathfrak{h})$)

– (R3) for $i=j$ is clear as $h_i = [e_i, f_i]$

(R3) for $i \neq j$ follows from $\alpha_i - \alpha_j \notin R \cup \{0\} \Rightarrow [e_i, f_j] = 0$

– To prove (R5) consider $\underbrace{\bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_{-\alpha_j + k\alpha_i}}_{V_{\alpha_i, -\alpha_j} \text{ - f.dim!}}$ which is an irreducible $\mathfrak{sl}(2,\mathbb{C})_{\alpha_i}$-module by [Lecture 16, Theorem 2]

But $ad(e_i) f_j = 0$ by (R3) $\Rightarrow f_j \in \mathfrak{g}_{-\alpha_j} \subset V_{\alpha_i, -\alpha_j}$ satisfies $ad(e_i) f_j = 0$, $ad(h_i) f_j = -a_{ij} f_j$

Thus, by $\mathfrak{sl}_2$-theory [Lecture 7], $ad(f_i)^{1-a_{ij}} f_j = 0 \Rightarrow$ (R5)

– The proof of (R4) is analogous (or use the opposite polarization)

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Let $R \subset E$ be any reduced root system. Pick a polarization $R = R_+ \cup R_-$, $\Pi = \{\alpha_1, \ldots, \alpha_r\}$ – simple roots

Def 1: Let $\mathfrak{g}(R)$ be the Lie algebra generated by $\{e_i, f_i, h_i\}_{i=1}^r$, with the defining rel-s (R1)–(R5) of Theorem 1.

Theorem 2 (Serre theorem)

1) The Lie algebra $\mathfrak{n}_+(R)$ of $\mathfrak{g}(R)$, generated by $\{e_i\}_{i=1}^r$, has the Serre rel-s $\mathrm{ad}(e_i)^{1-a_{ij}} e_j = 0$ as the defining rel-s. Similarly, the Lie subalgebra $\mathfrak{n}_-(R)$ of $\mathfrak{g}(R)$ generated by $\{f_i\}_{i=1}^r$ has the Serre rel-s $\mathrm{ad}(f_i)^{1-a_{ij}} f_j = 0$ as the defining rel-s. Finally, $\{h_i\}_{i=1}^r$ are linearly independent.

2) $\mathfrak{g}(R)$ is a sum of fin. dim-l $\mathfrak{sl}(2,\mathbb{C})_{\alpha_i} = \langle e_i, f_i, h_i \rangle$ -modules

3) $\mathfrak{g}(R)$ is fin. dim-l

4) $\mathfrak{g}(R)$ is semisimple and has root system R

As an immediate corollary of Theorems 1, 2, we get:

Corollary 1: a) If $\mathfrak{g}$ is a semisimple $\mathbb{C}$ Lie algebra with a root system R, then there is a natural isomorphism $\mathfrak{g} \simeq \mathfrak{g}(R)$

b) There is a natural bijection

$$\left\{\begin{matrix}\text{finite dimensional } \mathbb{C} \\ \text{semisimple Lie algebras}\end{matrix}\right\} \longleftrightarrow \{\text{reduced root systems}\}$$

$$\cup$$

$$\left\{\begin{matrix}\text{finite dimensional } \mathbb{C} \\ \text{simple Lie algebras}\end{matrix}\right\} \longleftrightarrow \{\text{reduced irreducible root systems}\}$$

Finally, combining with Main Theorem of Lecture 22, we get

Corollary 2: Isomorphism classes of simple $\mathbb{C}$ fin. dimensional Lie algebras are in bijection with Dynkin diagrams

$A_n\ (n \geq 1)$, $B_n\ (n \geq 2)$, $C_n\ (n \geq 3)$, $D_n\ (n \geq 4)$, E_6, E_7, E_8, F_4, G_2

Proof of Theorem 2

▶ We shall assume that R is irreducible, since $\mathfrak{g}(R_1 \cup R_2) = \mathfrak{g}(R_1) \oplus \mathfrak{g}(R_2)$
[Exercise: verify the above equality.

1) The proof of this step uses the following standard strategy.

First, consider a much bigger Lie algebra $\tilde{\mathfrak{g}}(R)$ generated by $\{e_i, f_i, h_i\}_{i=1}^{r}$ with the defining relations (R1)–(R3)

$\boxed{\text{Claim 1 (easy exercise): } \tilde{\mathfrak{g}}(R) = \tilde{\mathfrak{n}}_-(R) \oplus \tilde{\mathfrak{h}}(R) \oplus \tilde{\mathfrak{n}}_+(R) \text{ as a vector space, where } \tilde{\mathfrak{n}}_+(R) \text{ is gen-d by } e_i\text{'s}, \tilde{\mathfrak{n}}_-(R) \text{ is generated by } f_i}$

(Hint: Use (R1)–(R3) to reorder terms
As per direct sum, use the $\mathbb{Z}$-grading with $\deg(e_i) = 1 = -\deg(f_i)$, $\deg(h_i) = 0$ and show that $\tilde{\mathfrak{n}}_+(R) = \tilde{\mathfrak{g}}(R)_{>0}$, $\tilde{\mathfrak{n}}_-(R) = \tilde{\mathfrak{g}}(R)_{<0}$)

Claim 2: 1) The Lie algebra $\tilde{\mathfrak{n}}_+(R)$ is a free Lie algebra in $\{e_i\}_{i=1}^{r}$
2) The Lie algebra $\tilde{\mathfrak{n}}_-(R)$ is a free Lie algebra in $\{f_i\}_{i=1}^{r}$
3) $\tilde{\mathfrak{h}}(R)$ has a basis $\{h_i\}_{i=1}^{r}$

(Exercise: Prove Claim 2.
Hint: Construct an action of $\tilde{\mathfrak{g}}(R)$ on $U(\underbrace{\mathbb{C}^r}_{\text{Basis}\{h_i\}_{i=1}^r} \ltimes \text{Free Lie Algebra}\{\bar{e}_i\}_{i=1}^{r}) \simeq \mathbb{C}[h_1, \dots, h_r] \ltimes \mathbb{C}\langle \bar{e}_1, \dots, \bar{e}_r\rangle$

Finally, consider the elements $S_{ij}^{+} := \mathrm{ad}(e_i)^{1-a_{ij}} e_j \in \tilde{\mathfrak{n}}_+(R)$
$S_{ij}^{-} := \mathrm{ad}(f_i)^{1-a_{ij}} f_j \in \tilde{\mathfrak{n}}_-(R)$.

$\boxed{\text{Claim 3: } [f_k, S_{ij}^{+}] = 0 = [e_k, S_{ij}^{-}] \quad \forall k \in \{1, \dots, r\}}$

(Exercise: Prove Claim 3
Hint: for $k \neq i$, use $[f_k, e_{\neq k}] = 0$, $[f_k, e_k] = -h_k$
for $k = i$, use the $\mathfrak{sl}_2$-theory for $\mathfrak{sl}(2, \mathbb{C})_{\alpha_i}$.

Thus: The ideal I of Serre rel-s $(S_{ij}^{+}, S_{ij}^{-} \mid i \neq j)$ in $\tilde{\mathfrak{g}}(R)$ can be written as

$$\boxed{I = I_- \oplus I_+, \text{ where } I_\pm = (S_{ij}^{\pm} \mid i \neq j) \subset \tilde{\mathfrak{n}}_\pm(R)}$$

Combining this with Claim 2 completes the proof of 1).

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(Continuation of the proof)

2) Due to Serre rel-s, any of the generators $\{e_j, f_j, h_j\}_{j=1}^{r}$ generates a fin. dim-l $sl(2,\mathbb{C})_{\alpha_i}$-submodule. On the other hand, $\mathfrak{g}(R)$ is gen-d by these elements. Hence, the result follows from the following:

> **Claim 4** (easy exercise): If $a, b \in \tilde{\mathfrak{g}}(R)$ generate fin. dim-l $sl(2,\mathbb{C})_{\alpha_i}$-submodules, then so does $[a,b]$

3) The algebra $\mathfrak{g}(R)$ is $Q := \bigoplus_{i=1}^{r} \mathbb{Z}\alpha_i$-graded with $\deg(e_i) = \alpha_i = -\deg(f_i)$, $\deg(h_i) = 0$.

So: $\boxed{\mathfrak{g}(R) \simeq \bigoplus_{\alpha \in Q} \mathfrak{g}(R)_\alpha}$ Clear: $\mathfrak{g}(R)_\alpha = 0$ unless $\pm\alpha \in \bigoplus_{i=1}^{r} \mathbb{Z}_{\geq 0}\alpha_i$

On the other hand, $\dim \underbrace{\mathfrak{g}(R)_{\pm\alpha}}_{\subset \mathfrak{n}_\pm(R)} < \infty$ $\forall \alpha \in \bigoplus_{i=1}^{r} \mathbb{Z}_{\geq 0}\alpha_i$ for obvious reasons

It remains to show that $\#\{\alpha \mid \mathfrak{g}(R)_\alpha \neq 0\} < \infty$. In fact, we have:

> **Claim 5**: $\mathfrak{g}(R)_\alpha = 0$ if $\alpha \notin R \cup \{0\}$

The proof is by induction on $|ht(\alpha)|$. Assume $\alpha \in \bigoplus_{j=1}^{r} \mathbb{Z}_{\geq 0}\alpha_j$. If $\alpha = c\alpha_j$, then $\mathfrak{g}(R)_\alpha = \begin{cases} \mathbb{C}e_j, & c=1 \\ 0, & c>1. \end{cases}$ Otherwise, as in Thm 1, $\exists\, i$ s.t. $(\alpha, \alpha_i) > 0$.

Using part 2) and sl_2-theory: $\mathfrak{g}(R)_{s_i(\alpha)} \neq 0 \Rightarrow s_i(\alpha) \in R \cup \{0\}$ by induction hypothesis. But $R \cup \{0\}$ is W-invariant. Therefore, $\alpha \in R \cup \{0\}$.

Thus: $\mathfrak{g}(R)$ is fin. dimensional.

4) As $R = W(\Pi)$ and $\dim \mathfrak{g}(R)_{\alpha_i} = 1$, we see that $\mathfrak{g}(R) = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \underbrace{\mathfrak{g}(R)_\alpha}_{\text{1-dim subspaces}}$

If I is a nonzero ideal, then:

> **Claim 6** (easy exercise): I contains $\mathfrak{g}(R)_\alpha$ for some $\alpha \neq 0$

Again using W-invariance of weights and $R = W(\Pi)$, we conclude $\mathfrak{g}(R)_{\alpha_j} \subseteq I$

Commuting with f_j one/two times, we conclude $e_j, f_j, h_j \in I$. Now, if $j \frac{\quad}{\text{or multiple}} i$ in Dynkin diagram, then we also conclude $e_i, f_i, h_i \in I$.

As R-irreducible $\Rightarrow$ Dynkin diagram - connected $\Rightarrow e_i, f_i, h_i \in I\ \forall i \Rightarrow I = \mathfrak{g}(R)$.

So: $\mathfrak{g}(R)$ is simple. It's root system is clearly R