Lecture #24

Goal for this week: Develop the theory of finite-dimensional representations over $\mathbb{C}$ semisimple Lie algebras $\mathfrak{g}$ (by [Lecture 7, Prop 1] they can be exponentiated to give a repr-n of the connected simply connected Lie gp G with $\mathfrak{g} = \mathrm{Lie}(G)$)

Recall that every fin. dim-l $\mathfrak{g}$-module V is completely reducible (which was a consequence of Whitehead thm, see [Lect 13, Thm 2]). Therefore, we aim at:

$\boxed{\text{Classification of irreducible finite-dimensional } \mathfrak{g}\text{-modules}}$

Def 1: Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, $\lambda \in \mathfrak{h}^*$, and V be a $\mathfrak{g}$-module. Then $v \in V$ has a weight λ if $h(v) = \lambda(h)\cdot v \ \forall h \in \mathfrak{h}$, v is called a weight vector. Let $V[\lambda] := \{v \in V \mid h(v) = \lambda(h) v \ \forall h \in \mathfrak{h}\}$ – weight subspace of V of weight λ. If $V[\lambda] \neq 0$, then λ is called a weight of V. Set $P(V) = \{\lambda \in \mathfrak{h}^* \mid V[\lambda] \neq 0\}$

set of weights of V

The following properties are obvious (simple exercise to check at home):
1) $\mathfrak{g}_\alpha V[\lambda] \subseteq V[\lambda + \alpha] \quad \forall \lambda \in \mathfrak{h}^*, \alpha \in R \cup \{0\}$
2) vectors of different weights are lin. independent

Given a $\mathfrak{g}$-module V, set $V' = \mathrm{span}\{\text{all weight vectors}\} = \bigoplus_{\lambda \in \mathfrak{h}^*} V[\lambda]$.

Def 2: A $\mathfrak{g}$-module V has a weight decomposition (w.r.t. $\mathfrak{h} \subseteq \mathfrak{g}$) if $V' = V$

Exercise: Provide an example of ∞-dim $\mathfrak{g}$-module V which doesn't have weight decomp

Lemma 1: Any fin.-dim. $\mathfrak{g}$-module V has a weight decomposition. Furthermore, all weights of V are integral i.e. $\lambda(h_i) \in \mathbb{Z}$ $\forall \lambda \in P(V)$ $\forall i$

Consider $\mathfrak{sl}(2,\mathbb{C})_{\alpha_i} \subseteq \mathfrak{g}$. Viewing V as a fin. dim. module over $\mathfrak{sl}(2,\mathbb{C})_{\alpha_i}$, we see that h_i acts semisimply with integer eigenvalues. But $\{h_i\}_{i=1}^{rk(\mathfrak{g})}$ is a basis of $\mathfrak{h}$, hence, $\mathfrak{h} \curvearrowright V$ semisimply $\Rightarrow V = \bigoplus_{\lambda \in \mathfrak{h}^*} V[\lambda]$ ▯

So: V - fin. dim. $\mathfrak{g}$-module $\Rightarrow P(V)$ - finite subset of $P \subseteq \mathfrak{h}^*$

weight lattice, see Lecture #20

Def 3: A vector $\overset{\neq 0}{v} \in V$ is called a <u>highest weight vector of weight $\lambda \in \mathfrak{h}^*$</u> if
$$\begin{cases} e_i(v)=0 \ \forall i, \text{ equivalently } \mathfrak{n}_+(v)=0 \\ h(v)=\lambda(h)\cdot v \ \forall h\in\mathfrak{h}, \text{ equivalently } v\in V[\lambda]. \end{cases}$$
A $\mathfrak{g}$-module V is called a <u>highest weight representation with highest weight λ</u> if it is generated by such a vector.

Lemma 2: Any finite-dimensional $\mathfrak{g}$-representation $V\neq 0$ contains a nonzero highest weight vector of some weight $\lambda\in P$

▷ According to Lemma 1, we have $V=\bigoplus_{\lambda\in P(V)} V[\lambda]$ with $P(V)\subseteq P$ – finite set. As noted after Def 1, we have $e_i(V[\lambda])\subseteq V[\lambda+\alpha_i]$. But then picking $\lambda_{max}\in P(V)$ maximal in the sense of the pairing $\lambda(\rho^\vee)\in\mathbb{Z}$, we get $e_i(V[\lambda_{max}])=0 \ \forall i$, hence any $v\in V[\lambda_{max}]$ is a highest weight vector. ■

Corollary 1: Any irreducible fin. dim. $\mathfrak{g}$-module is a highest weight representation

▷ Follows from Lemma 2, since V is generated by any nonzero vector ■

We shall now take some detour and consider highest weight modules which are not necessarily fin. dim. In fact, the largest of those, so-called Verma modules, are key to the theory of finite-dimensional $\mathfrak{g}$-representations.

Def 4: Let $I_\lambda\subseteq U(\mathfrak{g})$ be the left ideal generated by $\{e_i\}_{i=1}^r\cup\{h-\lambda(h)\mid h\in\mathfrak{h}\}$. The <u>Verma module</u> M_λ is the quotient $U(\mathfrak{g})/I_\lambda$, viewed as a $\mathfrak{g}$-module.

Remark: Alternatively, evoking $\mathfrak{g}=\mathfrak{n}_-\oplus\mathfrak{h}\oplus\mathfrak{n}_+$, we can define M_λ as the <u>induced module</u> $U(\mathfrak{g})\otimes_{U(\mathfrak{h}\oplus\mathfrak{n}_+)}\mathbb{C}_\lambda =: \mathrm{Ind}_{U(\mathfrak{b}_+)}^{U(\mathfrak{g})}(\mathbb{C}_\lambda)$, where $\mathbb{C}_\lambda$ denotes a 1-dim representation of $\underbrace{\mathfrak{h}\oplus\mathfrak{n}_+}_{=:\mathfrak{b}_+}$ (hence $U(\mathfrak{b}_+)$) with $\mathfrak{n}_+$ acting by zero, $\mathfrak{h}$ acting via λ.

! For $\mathfrak{g}=\mathfrak{sl}_2$, we already encountered Verma modules M_λ back in Lecture 9, where it was shown that M_λ is irreducible unless $\lambda\in\mathbb{Z}_{\geq 0}$, and in the latter case $M_\lambda/M_{-\lambda-2}\simeq V_\lambda$ – fin. dim. $\mathfrak{sl}_2$-module.

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In fact, as a module over $U(\mathfrak{n}_-) \subseteq U(\mathfrak{g})$ Verma modules are free rank 1 modules:

Proposition 1: The map $U(\mathfrak{n}_-) \xrightarrow{\phi} M_\lambda$ given by $\phi(x) = x(\underbrace{v_\lambda}_{\text{class of } 1 \in U(\mathfrak{g})})$ is an isomorphism of left $U(\mathfrak{n}_-)$-modules

▶ Consider the vector space decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+ = \mathfrak{n}_- \oplus \mathfrak{b}_+$
PBW theorem implies that the multiplication map $U(\mathfrak{n}_-) \otimes U(\mathfrak{b}_+) \xrightarrow[m]{} U(\mathfrak{g})$ is a vector space isomorphism, see [Lecture 9, Corollary 2].

Consider $\varphi_\lambda : U(\mathfrak{b}_+) \longrightarrow \mathbb{C}$, given by $\varphi_\lambda(h) = \lambda(h)$, $\varphi_\lambda(e_i) = 0$, and set $K_\lambda = \mathrm{Ker}\, \varphi_\lambda$.

Then: $m^{-1}(I_\lambda) = U(\mathfrak{n}_-) \otimes K_\lambda$

Hence: $M_\lambda = U(\mathfrak{g})/I_\lambda \xleftarrow{\sim} U(\mathfrak{n}_-) \otimes U(\mathfrak{b}^+) / U(\mathfrak{n}_-) \otimes K_\lambda \simeq U(\mathfrak{n}_-)$ ∎

As an immediate corollary of the above result, we get:

Corollary 2: M_λ has a weight decomposition with $P(M_\lambda) = \{\lambda - \sum_{i=1}^{r} n_i \alpha_i \mid n_i \in \mathbb{Z}_{\geq 0}\}$
Moreover, $\dim M_\lambda[\lambda] = 1$ and $\dim M_\lambda[\mu] < \infty \ \forall \mu$

Notation: $Q_+ = \{\sum_{i=1}^{r} n_i \alpha_i \mid n_i \in \mathbb{Z}_{\geq 0}\}$, so that we get $P(M_\lambda) = \lambda - Q_+$

In fact, the Verma modules can be characterized by their universal property:

Proposition 2: If V is a $\mathfrak{g}$-module and $v \in V[\lambda]$ is a highest weight vector, then there is a unique $\mathfrak{g}$-module homomorphism $\begin{matrix} M_\lambda & \xrightarrow{\phi_v} & V \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ v_\lambda & \longmapsto & v \end{matrix}$

In particular, if V is a highest weight representation with highest weight λ, then it is a quotient of M_λ.

▶ To construct ϕ_v, note that we have $U(\mathfrak{g}) \to V$ given by $x \mapsto x(v)$, which contains I_λ in its kernel, hence gives rise to the desired $\phi_v : M_\lambda \longrightarrow V$.
The uniqueness is clear as M_λ is generated by v_λ.

Finally, if V is generated by such v, then we get $M_\lambda \twoheadrightarrow V$ ∎

Exercise: A quotient of any module with a highest weight decomposition must have a weight space decomposition

Corollary 3: Any highest weight $\mathfrak{g}$-module has a weight decomposition with fin. dim. weight subspace

In what follows, we shall use the partial order $\leq$ on $\mathfrak{h}^*$:

$$\boxed{\lambda \leq \mu \iff \mu - \lambda \in Q_+}$$

Lemma 3: Any highest weight $\mathfrak{g}$-module V has a unique highest weight and unique up to scaling highest weight vector

▶ Indeed, if λ is a highest weight of V, then $M_\lambda \twoheadrightarrow V$ and so $P(V) \subseteq P(M_\lambda)$ $\Rightarrow \forall \mu \in P(V)$ we have $\mu \leq \lambda$. Therefore, if μ is a highest weight of V too, then $\mu \leq \lambda$ & $\lambda \leq \mu \Rightarrow \lambda = \mu$.

Finally, $\dim M_\lambda[\lambda] = 1 \Rightarrow \dim V[\lambda] = 1 \Rightarrow$ highest weight vector is unique up to scaling ▣

Proposition 3: For any $\lambda \in \mathfrak{h}^*$, the Verma module M_λ has a unique irreducible quotient L_λ. Furthermore, L_λ is a quotient of any highest weight $\mathfrak{g}$-module V with highest weight λ

▶ Consider any $\mathfrak{g}$-submodule $W \subsetneq M_\lambda$. By the previous exercise, W has a weight decomposition $W = \bigoplus_{\mu \in \lambda - Q_+} W[\mu]$. But $W[\lambda] = 0$ as otherwise $v_\lambda \in W \Rightarrow W = M_\lambda$.

Define $J_\lambda := \sum_{W \subsetneq M_\lambda\text{-submodule}} W$ = the (non-direct) sum of all proper $\mathfrak{g}$-submodules of M_λ.

<u>Then</u>: J_λ is also a proper submodule, namely the largest one.

We set $L_\lambda := M_\lambda / J_\lambda$. First, we claim it is an irreducible $\mathfrak{g}$-module. Indeed, if $0 \subsetneq N_\lambda \subsetneq J_\lambda$, then preimage of N_λ under $\pi: M_\lambda \twoheadrightarrow L_\lambda$ would be a proper submodule strictly larger than J_λ, a contradiction. Second, given any highest weight $\mathfrak{g}$-module V of highest weight λ, we have by Prop 2: $\phi_v: M_\lambda \twoheadrightarrow V$. Then $\mathrm{Ker}(\phi_v)$ is a proper $\mathfrak{g}$-submodule of $M_\lambda \Rightarrow \mathrm{Ker}(\phi_v) \subseteq J_\lambda$, hence the epimorphism $M_\lambda \twoheadrightarrow L_\lambda$ descends through $V \twoheadrightarrow L_\lambda$ ▣

As an immediate corollary, we get:

Corollary 4: Irreducible highest weight $\mathfrak{g}$-modules are classified by $\lambda \in \mathfrak{h}^*$ via $\mathfrak{h}^* \ni \lambda \rightsquigarrow L_\lambda$ – from Prop 3

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As every irreducible finite-dimensional representation of $\mathfrak{g}$ is highest weight (Corollary 1), our original problem is equivalent to:

$$\boxed{\text{classification of } \lambda \in \mathfrak{h}^* \text{ such that } \dim(L_\lambda) < \infty}$$

Def 5: A weight $\lambda \in \mathfrak{h}^*$ is called dominant integral if $\underbrace{\alpha_i^\vee(\lambda)}_{=\lambda(h_i)} \in \mathbb{Z}_{\geq 0}$ $\forall i$.
$P_+ = \{\text{all dominant integral weights}\}$

Lemma 4: If L_λ is finite-dimensional, then $\lambda \in P_+$

▶ The vector $v_\lambda \in L_\lambda$ is the highest weight vector for $sl(2,\mathbb{C})_{\alpha_i}$ with the highest weight $\alpha_i^\vee(\lambda) = \lambda(h_i)$. Since L_λ - fin. dim, the $sl(2,\mathbb{C})_{\alpha_i}$-submodule generated by v_λ is also.
So: $\lambda \in P_+$ ∎

Lemma 5: If $\lambda \in P_+$, then we have $f_i^{\lambda(h_i)+1} v_\lambda = 0$ in L_λ

▶ By the sl_2-theory, applied to $sl(2,\mathbb{C})_{\alpha_i}$, we have $e_i f_i^{\lambda(h_i)+1} v_\lambda = 0$.
Also, we have $e_j f_i^{\lambda(h_i)+1} v_\lambda = f_i^{\lambda(h_i)+1} e_j v_\lambda = 0$ for $j \neq i$, as $[e_j, f_i] = 0$.
Finally, $f_i^{\lambda(h_i)+1} v_\lambda \in L_\lambda[\lambda - (\lambda(h_i)+1)\alpha_i]$.
Thus: the vector $f_i^{\lambda(h_i)+1} v_\lambda$ is a highest weight vector in L_λ. Therefore, it generates a proper submodule of L_λ (for degree reasons it cannot contain v_λ)
But: L_λ - irreducible $\Rightarrow$ this submodule is zero $\Rightarrow f_i^{\lambda(h_i)+1} v_\lambda = 0$ ∎

Exercise: Verify that the corresponding $\mathfrak{g}$-module homomorphism of Verma modules $M_{\lambda - (\lambda(h_i)+1)\alpha_i} \longrightarrow M_\lambda$ is injective

Lemma 6: Let V be a $\mathfrak{g}$-module with weight decomposition, with all $V[\lambda]$ being fin. dim. If V is a sum of fin. dim. $sl(2,\mathbb{C})_{\alpha_i}$-modules $\forall i$, then $\forall \lambda \in P, w \in W$, we have

$$\dim V[\lambda] = \dim V[w(\lambda)]$$

In particular, $P(V)$ is a W-invariant set.

Proof of Lemma 6

▸ As W is generated by simple reflections, it suffices to show $\dim V[\lambda] = \dim V[s_i\lambda]$. This follows immediately from $\mathfrak{sl}_2$-theory. Indeed, if $k_i := \lambda(h_i) \geq 0$, then

$$f_i^{k_i} : V[\lambda] \to V[s_i(\lambda)] \quad , \quad e_i^{k_i} : V[s_i(\lambda)] \to V[\lambda]$$

are injective. If $k_i < 0$, then $(s_i(\lambda))(h_i) > 0$, hence replace λ by $s_i(\lambda)$ above ∎

Theorem 1: $\forall \lambda \in P_+$, L_λ is finite-dimensional.

Therefore, we get a bijection

$$P_+ = \{\text{dominant integral weights}\} \xleftrightarrow{\text{1-to-1}} \left\{\begin{matrix}\text{irreducible} \\ \text{f.d. } \mathfrak{g}\text{-modules}\end{matrix}\right\}$$

$$\lambda \longmapsto L_\lambda$$

Moreover, $\forall \mu \in P$ $\forall w \in W$ we have $\dim L_\lambda[\mu] = \dim L_\lambda[w(\mu)]$

▸ By Lemma 5, $v_\lambda \in L_\lambda$ generates a finite-dimensional $\mathfrak{sl}(2,\mathbb{C})_{\alpha_i}$-module $\forall 1 \leq i \leq r = \mathrm{rk}(\mathfrak{g})$. We claim that then any vector of L_λ generates a fin. dim. $\mathfrak{sl}(2,\mathbb{C})_{\alpha_i}$-module. This can be proved by induction. Indeed, if

$$\underset{\text{f.dim } \mathfrak{sl}(2,\mathbb{C})_{\alpha_i}\text{-mod}}{v \in W \subset L_\lambda}, \text{ then } \forall x \in \mathfrak{g} \quad x.v \in \underbrace{\mathfrak{g}W}_{\text{f.dim as } \dim(\mathfrak{g}), \dim(W) < \infty} \subseteq L_\lambda \quad \text{and}$$

$\mathfrak{g}W$ is an $\mathfrak{sl}(2,\mathbb{C})_{\alpha_i}$-submodule, as

$$t.(y.w) = \underbrace{y.(tw)}_{\in \mathfrak{g}W} + \underbrace{[t,y].w}_{\in \mathfrak{g}W} \qquad \forall w \in W, y \in \mathfrak{g}, t \in \mathfrak{sl}(2,\mathbb{C})_{\alpha_i}$$

Then: $P(L_\lambda)$ is W-invariant by Lemma 6, and all weight subspaces are f.d.

The rest now follows from the following exercise:

Exercise: 1) $\forall \lambda \in P$, its Weyl group orbit contains exactly one element of P_+

2) $\forall \lambda \in P$, the intersection $P_+ \cap (\lambda - Q_+)$ is finite.

Indeed, the above exercise $\Rightarrow$ $P(L_\lambda)$ is finite $\Rightarrow$ $L_\lambda = \bigoplus_{\mu \in P(L_\lambda)} L_\lambda[\mu]$ is fin. dimensional. The rest follows from Corollaries 1+4 ∎