Lecture #25-26

Let V be a finite-dimensional representation of a semisimple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$. If G is the corresponding connected simply connected Lie group with $\mathrm{Lie}(G)=\mathfrak{g}$, then

$$\mathfrak{g} \curvearrowright V \underset{\text{exponentiated}}{\rightsquigarrow} G \curvearrowright V.$$

As we know from the case of representations of finite gps, an important quantity of above is:

$$\boxed{\text{character } \chi_V: G \to \mathbb{C} \quad \text{given by } \chi_V(g)=\mathrm{tr}_V(g)}$$

As χ_V is conjugacy-class invariant, the values of $\{\chi_V(e^h) \mid h\in\mathfrak{h}\}$ determine $\{\chi_V(e^x) \mid x\text{-s.s}\}$. On the other hand, the set of s.s (semisimple) elements is dense in $\mathfrak{g}$.

[Rem: One can actually appeal to "strongly regular elements" $\subsetneq$ semisimple elements, Lect 17

Since χ_V is an analytic function on G, it is determined by its values on nonempty open set $\{e^x \mid x\text{-s.s.}, \|x\|\ll 1\} \subseteq$ nbhd of $1\in G$ which is $\simeq$ nbhd of $0\in\mathfrak{g}$ (see Lecture 5).

$$\boxed{\underline{\text{SO}}: \chi_V: G\to\mathbb{C} \text{ is determined by the values } \{\chi_V(e^h) \mid h\in\mathfrak{h}\}}$$

However, as noted last time, V has a weight decomposition $V=\bigoplus_{\lambda\in P} V[\lambda]$ and by definition, we have

$$\boxed{\chi_V(e^h) = \sum_{\lambda\in P} \dim V[\lambda]\cdot e^{\lambda(h)} = \sum_{\lambda\in \underbrace{P(V)}_{\text{finite set}}} \dim V[\lambda]\cdot e^{\lambda(h)}}$$

Thus all the information is readily captured by the following invariant that "counts" dimensions of weight components

Def 1: Character of V is: $\boxed{\chi_V := \sum_{\mu\in P} \dim V[\mu]\cdot e^{\mu} \ \in \mathbb{Z}[P]}$

Here, $\mathbb{Z}[P]$ is the group algebra of the abelian gp P, i.e. it's a vector space with basis $\{e^\lambda \mid \lambda\in P\}$ and product $e^\lambda\cdot e^\mu = e^{\lambda+\mu}$.

[Rem: e^λ can be viewed as an analytic function on $\mathfrak{h}$ via $e^\lambda(h):=e^{\lambda(h)}$

Thus, we may denote e^0 simply by 1.

[Rem: As $P\simeq \bigoplus_{i=1}^{r} \mathbb{Z}\omega_i$, ω_i - fundamental weights, we have $\mathbb{Z}[P]\simeq \mathbb{Z}[x_1^{\pm1},\ldots,x_r^{\pm1}]$, $x_i:=e^{\omega_i}$

Note that for $\mathfrak{g}=\mathfrak{sl}_2$, we have already encountered characters in Lecture 7, where they were defined via $\chi_V(z)=\sum_{m\in\mathbb{Z}} \dim V(m)\cdot z^m$. As $P=\mathbb{Z}\omega_1$ for $\mathfrak{sl}_2$, if we match $z \leftrightarrow e^{\omega_1}$, then it precisely recovers the above notion.

Lecture #25

Since we noted in Lecture #24 that one really needs more general repr-s, not just finite-dimensional ones, to develop the theory of the latter, we shall now extend the character to a bigger category of $\mathfrak{g}$-modules:

Def 2: The category $\mathcal{O}$ is the category of representation V of $\mathfrak{g}$ which admit weight decomposition into fin.dim'l weight spaces

$$\boxed{V = \bigoplus_{\mu\in P} V[\mu], \quad \dim V[\mu] < \infty}$$

s.t. $\exists$ finite set $\lambda_1, \dots, \lambda_m \in P$ satisfying

$$\boxed{P(V) \subseteq \bigcup_{1\le i\le m} (\lambda_i - Q_+)}$$

[Rem: Usually, $\lambda_i \in \mathfrak{h}^*$, but for our purpose it suffices to consider $\lambda_i \in P$.

[Example: Any highest weight module is in $\mathcal{O}$ (recall $P(M_\lambda) = \lambda - Q_+$)

We would like now to generalize character χ_V to $V\in\mathcal{O}$. As $P(V)$ may be infinite, we will get a certain completion of $\mathbb{Z}[P]$. Explicitly, let

$$\boxed{R \text{ be the ring of } \left\{ \sum_{\mu\in P} c_\mu \cdot e^\mu \,\middle|\, c_\mu \in \mathbb{Z} \ \&\ \{\mu \mid c_\mu \neq 0\} \subseteq \bigcup (\lambda_i - Q_+) \text{ for some } \lambda_1, \dots, \lambda_m \in P \right\}}$$

(**Exercise (easy)**: Verify that this is indeed a ring

Then for any $V\in\mathcal{O}$, we define its character χ_V as before:

$$\boxed{\chi_V := \sum_{\mu\in P} \dim V[\mu]\, e^\mu \in R}$$

Lemma 1: The following properties hold:

a) If $U \simeq V \oplus W$ in $\mathcal{O}$, then $\chi_U = \chi_V + \chi_W$.

b) More generally, if $0 \to V \to U \to W$ is a short exact sequence in $\mathcal{O}$, then $\chi_U = \chi_V + \chi_W$.

c) For any $V, W \in \mathcal{O}$, we have $V\otimes W \in \mathcal{O}$ and $\chi_{V\otimes W} = \chi_V \cdot \chi_W$

(**Exercise (easy)**: prove this lemma

Lecture #25

Lemma 2: For any $\lambda \in P$, the character of the Verma module M_λ is given by:

$$\chi_{M_\lambda} = \frac{e^\lambda}{\prod_{\alpha\in R_+}(1-e^{-\alpha})} \quad \text{where} \quad \frac{1}{1-e^{-\alpha}} = 1+e^{-\alpha}+e^{-2\alpha}+\dots$$

▶ Accordingly to [Lect 24, Prop 1], we have $U(\mathfrak{n}_-) \xrightarrow{\sim} M_\lambda$ given by $x \mapsto x(v_\lambda)$.

But accordingly to PBW thm, monomials $\{\prod_{\alpha\in R_+} f_\alpha^{n_\alpha} \mid n_\alpha \in \mathbb{Z}_{\geq 0}\}$ form a basis of $U(\mathfrak{n}_-)$

Thus: $\chi_{M_\lambda} = e^\lambda \sum_{\mu \in Q_+} e^{-\mu}\cdot \underbrace{P(\mu)}_{\text{Kostant partition function}}$, $P(\mu) = \#\{\text{decompositions } \mu = \sum_{\alpha\in R_+} n_\alpha\cdot\alpha \text{ with } n_\alpha \in \mathbb{Z}_{\geq 0}\}$

But the latter is easily seen to be precisely $e^\lambda\cdot \prod_{\alpha\in R_+}\frac{1}{1-e^{-\alpha}}$ ∎

For the latter purpose, let's rewrite the above f-la. Recall $\rho = \frac{1}{2}\sum_{\alpha\in R_+}\alpha$. Then:

$$\chi_{M_\lambda} = \frac{e^{\lambda+\rho}}{\prod_{\alpha\in R_+}(e^{\alpha/2}-e^{-\alpha/2})} =: \frac{e^{\lambda+\rho}}{\underbrace{\Delta}_{\text{Weyl denominator}}}$$

Def 3: The sign character is a group homomorphism $\varepsilon: W \to \mathbb{Z}/2 \simeq \{\pm 1\}$ given by $\varepsilon(w) = \det(w|_{\mathfrak{h}^*}) = (-1)^{\ell(w)}$. Also, $f \in \mathbb{Z}[P]$ is anti-invariant if $w(f) = \varepsilon(w)\cdot f$ $\forall w\in W$

The following simple property of Δ is key to the rest:

Lemma 3: The Weyl denominator Δ is anti-invariant, i.e. $w(\Delta) = \varepsilon(w)\cdot\Delta$ $\forall w \in W$

▶ As W is generated by simple reflections s_i, it suffices to check for those.

But: $s_i(\alpha_i) = -\alpha_i$ and s_i permutes $R_+\setminus\{\alpha_i\}$, hence

$$s_i(\Delta) = (e^{-\alpha_i/2} - e^{\alpha_i/2})\cdot \prod_{\alpha\in R_+\setminus\{\alpha_i\}} (e^{\alpha/2}-e^{-\alpha/2}) = -\Delta = \varepsilon(s_i)\cdot\Delta$$ ∎

Now we are ready to state the key result of today's class:

Theorem 1 (Weyl character formula): For any $\lambda \in P_+$, the character χ_{L_λ} of the irreducible finite-dimensional $\mathfrak{g}$-module L_λ is given by

$$\boxed{\chi_{L_\lambda} = \frac{\sum_{w\in W}(-1)^{\ell(w)} e^{w(\lambda+\rho)}}{\Delta}}$$

Before presenting the proof, we shall first discuss some corollaries.

Corollary 1 (Weyl denominator formula): $\Delta = \sum_{w\in W} (-1)^{\ell(w)} \cdot e^{w\rho}$

▷ Apply Thm 1 to $\lambda=0$ and note that $\chi_{V_0}=1$. ∎

Corollary 2: For any $\lambda\in P_+$, we have
$$\chi_{L_\lambda} = \frac{\sum_{w\in W}(-1)^{\ell(w)} e^{w(\lambda+\rho)}}{\sum_{w\in W}(-1)^{\ell(w)} e^{w\rho}}$$

▷ Combine Thm 1 + Cor 1 ∎

The Weyl character formula also allows to compute $\dim(L_\lambda)=\chi_{L_\lambda}(e^0)$. However, the substitution into the Weyl char. f-la doesn't immediately provide the answer, as both numerator & denominator vanish at e^0. The standard way to resolve this indeterminancy is to first compute the q-character:

Def 4: For a finite-dimensional $\mathfrak{g}$-module V, define $\dim_q V\in\mathbb{Z}[q,q^{-1}]$ via
$$\boxed{\dim_q V = \mathrm{tr}_V(q^{2\rho}) = {\sum_{\lambda\in P}}' \dim V[\lambda]\cdot q^{(2\rho,\lambda)}}$$
where $(\cdot,\cdot)$ is rescaled to satisfy $(\lambda,\mu)\in\mathbb{Z}\ \forall \lambda,\mu\in P$.

Thus, $\dim_q V = \pi_\rho(\chi_V)$, where $\pi_\rho: \mathbb{Z}[P]\to\mathbb{Z}[q,q^{-1}]$, $e^\mu\mapsto q^{(2\rho,\mu)}$

Proposition 1: For $\lambda\in P_+$, we have
$$\dim_q L_\lambda = \prod_{\alpha\in R_+}\frac{q^{(\lambda+\rho,\alpha)}-q^{-(\lambda+\rho,\alpha)}}{q^{(\rho,\alpha)}-q^{-(\rho,\alpha)}}$$

▷ $$\dim_q L_\lambda = \pi_\rho(\chi_{L_\lambda}) \overset{\text{Thm 1}}{=} \frac{\sum_{w\in W}(-1)^{\ell(w)} q^{2(w(\lambda+\rho),\rho)}}{\prod_{\alpha\in R_+}(q^{(\alpha,\rho)}-q^{-(\alpha,\rho)})} \underset{\ell(w)=\ell(w^{-1})}{\overset{w\leftrightarrow w^{-1}}{=}} \frac{\sum_{w\in W}(-1)^{\ell(w)} q^{2(\lambda+\rho,w(\rho))}}{\prod_{\alpha\in R_+}(q^{(\rho,\alpha)}-q^{-(\rho,\alpha)})}$$
Here, the numerator can be rewritten using $\boxed{\pi_{\lambda+\rho}: \mathbb{Z}[P]\to\mathbb{Z}[q,q^{-1}],\ e^\mu\mapsto e^{(2(\lambda+\rho),\mu)}}$ as:
$$\sum_{w\in W}(-1)^{\ell(w)} q^{(2(\lambda+\rho),w(\rho))} = \pi_{\lambda+\rho}\Big(\sum_{w\in W}(-1)^{\ell(w)} e^{w\rho}\Big) \underset{\text{Cor 1}}{=} \pi_{\lambda+\rho}(\Delta) = \prod_{\alpha\in R_+}\left(q^{(\lambda+\rho,\alpha)}-q^{-(\lambda+\rho,\alpha)}\right)$$
This completes the proof ∎

As $\dim V = (\dim_q V)|_{q=1}$, we obtain by L'Hôpital rule the dimension formula:

Corollary 3: For $\lambda\in P_+$, we have $\dim L_\lambda = \dfrac{\prod_{\alpha\in R_+}(\lambda+\rho,\alpha)}{\prod_{\alpha\in R_+}(\rho,\alpha)}$

[Rmk: It's a-priori nonobvious that the right-hand side is an integer!

Lecture #25

Proof of Theorem 1

Let us finally prove the Weyl character formula.

- An important ingredient is again the Casimir element C, see [Lecture 13, Def 4]. Recall that $C = \sum_i a_i a^i \in U(\mathfrak{g})$, where $\{a_i\}, \{a^i\}$ – dual bases of $\mathfrak{g}$ w.r.t. Killing form $(,)$.

Let $\{x_j\}_{j=1}^r$ be an orthonormal basis of $\mathfrak{h}$ w.r.t. $(,)|_{\mathfrak{h}\times\mathfrak{h}}$. We also pick $\{e_\alpha, f_\alpha\}_{\alpha\in R_+}$ so that $(e_\alpha, f_\alpha) = \frac{2}{(\alpha,\alpha)}$, so that $\{e_\alpha, f_\alpha, h_\alpha = \frac{2}{(\alpha,\alpha)} H_\alpha\}$ satisfy sl_2-relations, see [Lecture 16, Lemma 2]. Then:

$$\boxed{C = \sum_{j=1}^r x_j^2 + \sum_{\alpha\in R_+} \frac{(\alpha,\alpha)}{2}(e_\alpha f_\alpha + f_\alpha e_\alpha) = \sum_{j=1}^r x_j^2 + \sum_{\alpha\in R_+} H_\alpha + \sum_{\alpha\in R_+} (\alpha,\alpha)\cdot f_\alpha e_\alpha}$$

Lemma 4: If V is a highest weight $\mathfrak{g}$-module with highest weight λ, then
$$C|_V = (\lambda, \lambda+2\rho)\cdot \mathrm{Id}_V = (|\lambda+\rho|^2 - |\rho|^2)\cdot \mathrm{Id}_V$$

Since Casimir elt is central (Lect 13, Lemma 4), and V is generated by its highest wt vector v_λ, it suffices to prove $C(v_\lambda) = (\lambda,\lambda+2\rho)\cdot v_\lambda$.

But:
1) $e_\alpha(v_\lambda) = 0 \ \forall \alpha\in R_+ \Rightarrow$ all $f_\alpha e_\alpha$ annihilate v_λ.

2) $\sum_{j=1}^r x_j^2(v_\lambda) = \sum_{j=1}^r \lambda(x_j)\cdot\lambda(x_j)\cdot v_\lambda = (\lambda,\lambda)\cdot v_\lambda$

3) $H_\alpha(v_\lambda) = \lambda(H_\alpha)\cdot v_\lambda = (\lambda,\alpha)\cdot v_\lambda \Rightarrow \sum_{\alpha\in R_+} H_\alpha(v_\lambda) = (\lambda, 2\rho)\cdot v_\lambda$

- Consider the product $\chi_{L_\lambda}\cdot\Delta \in \mathbb{Z}[P]$. Know that Δ is W-anti-invariant, while χ_{L_λ} is W-invariant ([Lect 24, Thm 1]), hence, $\chi_{L_\lambda}\cdot\Delta$ is W-anti-invariant.

So: $\boxed{\chi_{L_\lambda}\cdot\Delta = \sum_{\mu\in P} c_\mu\cdot e^\mu \quad \text{with } c_{w\mu} = (-1)^{\ell(w)} c_\mu}$ Also: $\boxed{c_\mu = 0 \text{ if } \mu\notin\lambda+\rho-Q_+}$

On the other hand, we know that $\forall \mu\in P$ its W-orbit contains a (unique!) element in P_+ (see [Homework 12, Problem 5(b)]). Thus it remains to prove:

$\boxed{\text{Claim 1: } \forall \mu\in P_+\cap(\lambda+\rho-Q_+), \text{ we have } c_\mu = 0 \text{ unless } \mu = \lambda+\rho}$

Lecture #25

(Continuation of the proof)

- $\boxed{\begin{array}{l}\text{Claim 2: For any } V\in\mathcal{O}\text{, we have } \chi_V=\sum_{\mu\in P} b_\mu\cdot\chi_{M_\mu} \text{ with } b_\mu\in\mathbb{Z} \text{ and} \\ \{\mu \mid b_\mu\neq 0\}\subset\bigcup_{\lambda\in P(V)}(\lambda-Q_+)\end{array}}$

▷ This is proved by constructing on each step a homomorphism from $\underbrace{\oplus M_?}_{\text{direct sum of Verma modules}}$ to one of the modules constructed in the previous step,

and taking Ker, Coker. The result follows from Lemma 1b). □

In our case, we get $\chi_{L_\lambda}=\sum_\mu b_\mu\cdot\chi_{M_\mu}$. But it follows from the above proof of Claim 2 that $C_{|M_\mu}=C_{|L_\lambda}$. In particular, $\left.\begin{array}{l} y_{L_\lambda}\cdot\Delta=\sum_\mu c_\mu\cdot e^\mu \\ \quad\parallel\ \sum_\mu b_\mu\cdot e^{\mu+\rho}\end{array}\right\}\Rightarrow c_\mu=b_{\mu-\rho}$.

- To establish Claim 1, it remains to show:

$$\boxed{\forall \mu\in P_+\cap(\lambda-Q_+),\ \mu\neq\lambda \text{ we have } \underbrace{C_{|M_\mu}}_{|\mu+\rho|^2-|\rho|^2}\neq\underbrace{C_{|L_\lambda}}_{|\lambda+\rho|^2-|\rho|^2}}$$

If $\mu=\lambda-\beta\in P_+$, $\beta\in Q_+\setminus\{0\}$, then:

$$|\lambda+\rho|^2-|\mu+\rho|^2=2(\lambda+\rho,\beta)-|\beta|^2=(\lambda+\rho,\beta)+(\lambda+\rho-\beta,\beta)=\overbrace{\underbrace{(\lambda+\rho,\beta)}_{>0}}^{=(\lambda,\beta)+(\rho,\beta)}+\underbrace{(\mu,\beta)}_{\geq 0}>0$$

This establishes Claim 1, hence, also Theorem 1 ∎