Lecture #27

- Last week:
  - classified all irreducible (hence all) finite-dimensional modules over semisimple $\mathfrak{g}$ (Lect 24)
  - established character formula for χ_{L_λ} (Lect 25-26)

- Today we are going to further comment on these both results.

Lemma 1: Let $\lambda \in P_+$ and M_λ be the Verma module of highest weight λ with $v_\lambda \overset{\neq 0}{\in} M_\lambda[\lambda]$. Then $\forall i$, the vector $f_i^{\lambda(h_i)+1}(v_\lambda)$ is a highest weight vector of M_λ. Moreover, the submodule M_i of M_λ generated by $f_i^{\lambda(h_i)+1} v_\lambda$ is isomorphic to Verma module $M_{\lambda-(\lambda(h_i)+1)\alpha_i}$

▶ The equality $e_j f_i^{\lambda(h_i)+1}(v_\lambda)=0 \ \forall j$ was already checked in [Lecture 24, Lemma 5] (even though loc.cit. was about L_λ not M_λ, the argument is the same). Also it's clear that $f_i^{\lambda(h_i)+1}(v_\lambda) \in M_\lambda[\lambda-(\lambda(h_i)+1)\alpha_i]$. Finally, $f_i^{\lambda(h_i)+1}(v_\lambda) \neq 0$ as $U(\mathfrak{n}_-) \xrightarrow{\sim} M_\lambda$ [Lecture 24 Prop 1]

Hence: there is a nonzero $\mathfrak{g}$-module homomorphism [Lecture 24 Proposition 2]

$$\boxed{M_{\lambda-(\lambda(h_i)+1)\alpha_i} \longrightarrow M_\lambda}$$
$$\text{ht.wt. vector } v_{\lambda-(\lambda(h_i)+1)\alpha_i} \longmapsto f_i^{\lambda(h_i)+1}(v_\lambda)$$

But by [Hwk 12, Problem 4], any nonzero $\mathfrak{g}$-module homomorphism between Verma modules is injective. Hence, $M_{\lambda-(\lambda(h_i)+1)\alpha_i} \xrightarrow{\sim} M_i \subsetneq M_\lambda$

generated by $f_i^{\lambda(h_i)+1}(v_\lambda)$ ■

For $\lambda \in P_+$ the result above provides a family of submodules $\{M_i\}_{i=1}^{r=\mathrm{rk}(\mathfrak{g})}$ of M_λ. Since each of them is proper, so is their (not direct!) sum $\sum M_i$. Define

$$\boxed{\tilde{L}_\lambda := M_\lambda / \sum_{i=1}^{r} M_i}$$

Theorem 1: a) $\tilde{L}_\lambda$ is finite-dimensional

b) $\tilde{L}_\lambda$ is irreducible $\mathfrak{g}$-module, hence $\tilde{L}_\lambda \simeq L_\lambda$

[Example 1: For $\mathfrak{g}=\mathfrak{sl}_2$, this reduces to $V_n \simeq M_n / M_{-n-2}$, established in Lecture 9.

Lecture #27

Proof of Theorem 1

b) If $\tilde{L}_\lambda$ is already shown to be fin. dim, with clearly $P(\tilde{L}_\lambda) \subseteq \lambda - Q_+$, we can decompose $\tilde{L}_\lambda$ into the direct sum of irreducibles:

$$\boxed{\tilde{L}_\lambda = \bigoplus_{\substack{\mu \leq \lambda \\ \mu \in P_+}} L_\mu^{\oplus m_\mu}}$$

As $\dim \tilde{L}_\lambda[\lambda] = 1$, we see that $m_\lambda = 1 \Rightarrow \tilde{L}_\lambda = L_\lambda \oplus \bigoplus_{\substack{\mu < \lambda \\ \mu \in P_+}} L_\mu^{\oplus m_\mu}$ with L_λ gen-d by $0 \neq v_\lambda \in \tilde{L}_\lambda[\lambda]$

But then on one hand v_λ generates all $\tilde{L}_\lambda$, while on the other hand - only L_λ.

Thus: all $m_\mu = 0$ for $\mu < \lambda$ AND $\tilde{L}_\lambda = L_\lambda$.

a) The proof of part a) saying that $\dim(\tilde{L}_\lambda) < \infty$ is actually analogous to that of [Lecture 24, Theorem 1].

- First, we note that $\forall v \in \tilde{L}_\lambda$ $\forall 1 \leq i \leq r$, the $sl(2,\mathbb{C})_{\alpha_i}$-submodule generated by v (given explicitly by $\mathcal{U}(sl(2,\mathbb{C})_{\alpha_i})v$) is finite-dimensional.
  The argument is the same as it was for L_λ. Namely, the result is true for $v_\lambda \in \tilde{L}_\lambda$ due to Lemma 1. Hence, it also holds for any $v \in \tilde{L}_\lambda$.

- B/c of the above, [Lecture 24, Lemma 6] applies, and we get:

$$\boxed{\tilde{L}_\lambda = \bigoplus_{\substack{\mu \leq \lambda \\ \mu \in P}} \tilde{L}_\lambda[\mu] \quad \text{with } \dim \tilde{L}_\lambda[\mu] = \dim \tilde{L}_\lambda[w\mu] \quad \forall \mu\ \forall w \in W}$$

Now using the same argument as in [Lecture 24, Thm 1], see [Hwk 12, Prob. 5], we get $P(\tilde{L}_\lambda)$ is a finite set. But weight subspaces $\tilde{L}_\lambda[\mu]$ are all fin. dimensional (as so they are in M_λ). This implies $\dim(\tilde{L}_\lambda) < \infty$ ∎

In fact, the condition from [Lecture 24, Lemma 6] is very important and deserves a special treatment as a definition.

Def 1: A $\mathfrak{g}$-module V is called integrable if $\forall v \in V$ $\forall 1 \leq i \leq r$, we have

$$\boxed{\dim(\mathcal{U}(sl(2,\mathbb{C}))_{\alpha_i} v) < \infty}$$

Lecture #27

Define the shifted (a.k.a. dot) action of the Weyl group W on $\mathfrak{h}^*$:

Def 2: $\boxed{w\circ\lambda := w(\lambda+\rho)-\rho}$ with $\rho=\frac{1}{2}\sum_{\alpha\in R_+}\alpha = \sum_{i=1}^{r}\omega_i$ ← check it's indeed an action!

Note: $s_i(\rho)=\rho-\alpha_i$ by [Lecture 21, Lemma 2)

$\Downarrow$

$s_i\circ\lambda = s_i(\lambda+\rho)-\rho = s_i(\lambda)-\alpha_i = \lambda-(\lambda(h_i)+1)\alpha_i$

In particular, $\boxed{f_i^{\lambda(h_i)+1}v_\lambda \in M_\lambda[s_i\circ\lambda]}$ in Lemma 1.

Thus, by Theorem 1, we have a short exact sequence $\bigoplus_i M_{s_i\cdot\lambda}\to M_\lambda\to L_\lambda\to 0$.

The following beautiful result was established by Bernstein-Gelfand-Gelfand:

Theorem 2 (BGG resolution): For $\lambda\in P_+$, there exists a long exact sequence

$$\boxed{0\to M_{w_0\circ\lambda}\to\cdots\to\bigoplus_{w\in W}^{\ell(w)=k} M_{w\circ\lambda}\to\cdots\to\bigoplus_{i=1}^{r} M_{s_i\cdot\lambda}\to M_\lambda\to L_\lambda\to 0}$$

called the BGG resolution of L_λ

where $\ell: W\to\mathbb{Z}_{\geq 0}$ is the length, $w_0\in W$ – the longest element

The proof is out of scope of the present course

Example 1: For $\mathfrak{g}=\mathfrak{sl}_3(\mathbb{C})$, $W=S_3=\{1, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1=s_2s_1s_2\}$, we get

$$0\to M_{s_1s_2s_1\cdot\lambda}\to (M_{s_1s_2\cdot\lambda}\oplus M_{s_2s_1\cdot\lambda})\to(M_{s_1\cdot\lambda}\oplus M_{s_2\circ\lambda})\to M_\lambda\to L_\lambda\to 0$$

Remark: Evaluating characters in the BGG resolution, we get

$$\boxed{\chi_{L_\lambda}=\sum_{w\in W}(-1)^{\ell(w)}\chi_{M_{w\circ\lambda}}=\sum_{w\in W}(-1)^{\ell(w)}\cdot\frac{e^{w(\lambda+\rho)-\rho}}{\prod_{\alpha\in R_+}(1-e^{-\alpha})}=\frac{\sum_{w\in W}(-1)^{\ell(w)}e^{w(\lambda+\rho)}}{\prod_{\alpha\in R_+}(e^{\alpha/2}-e^{-\alpha/2})}}$$

Therefore, BGG resolution can be thought of as a "categorification" of the Weyl character f-la.

We note that Thm 2 greatly refines Claim 2 from our proof of Theorem 1 in Lecture 25-26 (which only used a family of iterative short exact sequences)

Lecture #27

We end the lecture with a couple more observations.

Example 3: Let $\mathfrak{g}$ be a simple Lie algebra$/\mathbb{C}$, and consider its adjoint repr-n:

$$\text{ad}: \mathfrak{g} \curvearrowright \mathfrak{g} \quad \leftarrow \text{simple module as } \mathfrak{g} \text{ has no proper nonzero ideals}$$

Hence, by general theory, it is $\simeq L_\theta$ for some $\theta \in P$.

Note that the weight decomposition is precisely $\mathfrak{g} \simeq \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$.

Therefore: $\theta \in R$ is such that $\forall \mu \in R$ have $\mu \leq \theta$. In particular, $\theta + \alpha \notin R \ \forall \alpha \in R_+$. $\Rightarrow \begin{cases} \theta \in R_+ \\ ht(\mu) < ht(\theta) \ \forall \mu \in R \setminus \theta \end{cases}$

Def 3: This $\theta \in R_+$ is called the maximal or highest root of $\mathfrak{g}$.

The number $h = ht(\theta) + 1$ is called the Coxeter number of $\mathfrak{g}$.

[Exercise: Find the highest root of $sl_n(\mathbb{C})$]

Proposition 1: Characters $\{\chi_{L_\lambda}\}_{\lambda \in P_+}$ form a basis in the algebra $\mathbb{C}[P]^W$.

► For $\lambda \in P_+$, define

$$\boxed{m_\lambda := \sum_{\mu \in W\lambda} e^\mu} \quad - \text{"averaging over the orbit"}.$$

Clearly $\{m_\lambda\}_{\lambda \in P_+}$ form a basis of $\mathbb{C}[P]^W$ (see [Hwk 12, Problem 5])

On the other hand, $\chi_{L_\lambda} \in \mathbb{C}[P]^W$ and has the form $\boxed{\chi_{L_\lambda} = m_\lambda + \sum_{\substack{\mu < \lambda \\ \mu \in P_+}} c_\lambda^\mu m_\mu}$ for some $c_\mu^\lambda \in \mathbb{Z}$. Therefore, the change of basis matrix expressing $\{\chi_{L_\lambda}\}$ via $\{m_\mu\}$ is upper-triangular with 1's on the main diagonal.

This implies the result! ∎

Lecture #27

Let us see what the results of Lecture 25-26 look like in the simplest case of $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ (i.e. type A_{n-1}). In this case, the root system is:

$$\boxed{R = \{e_i - e_j \mid i \neq j\} \subset \mathfrak{h}^*_{\mathbb{R}} = \mathbb{R}^n / \mathbb{R}(1, \ldots, 1)}$$

For the polarization given by $t = (t_1, \ldots, t_n) \in \mathbb{R}^n$ with $t_1 > t_2 > \ldots > t_n$, have:

$$\boxed{R_+ = \{e_i - e_j \mid i < j\} \text{ and simple roots } \alpha_i = e_i - e_{i+1} \ (1 \leq i \leq n-1)}$$

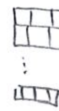
Moreover, the weight lattice P and the set $P_+ \subset P$ of dominant integral weights are:

$$\boxed{P = \{(\lambda_1, \ldots, \lambda_n) \in \mathfrak{h}^*_{\mathbb{R}} \mid \lambda_i - \lambda_j \in \mathbb{Z}\} \supset P_+ = \{(\lambda_1, \ldots, \lambda_n) \in \mathfrak{h}^* \mid \lambda_i - \lambda_{i+1} \in \mathbb{Z}_{\geq 0}\}}$$

Since adding a multiple of $(1, \ldots, 1)$ does not change the weight, we get:

$$\boxed{P_+ = \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n \mid \lambda_1 \geq \ldots \geq \lambda_n\} / \mathbb{Z}(1, \ldots, 1) = \{(\lambda_1, \ldots, \lambda_{n-1}, 0) \in \mathbb{Z}^n_{\geq 0} \mid \lambda_1 \geq \ldots \geq \lambda_{n-1} \geq 0\}}$$

We can represent $(\lambda_1, \ldots, \lambda_{n-1}, 0)$ by a Young diagram



Example 4: Based on [Homework 12, Problem 6], we have:

a) $S^k\mathbb{C}^n$ is an irreducible highest weight $\mathfrak{sl}_n$-module with highest weight $= (k, 0, \ldots, 0, 0)$ for any $k \geq 1$.

b) $\Lambda^k\mathbb{C}^n$ is an irreducible highest weight $\mathfrak{sl}_n$-module with highest weight $= (\underbrace{1, \ldots, 1}_{k}, 0, \ldots, 0)$ for $1 \leq k \leq n$

We can also identify $\mathbb{Z}[P] \simeq \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] / (x_1 \cdots x_n - 1)$ with $x_i = e^{(0, \ldots, \overset{i^{th} \text{ spot}}{1}, 0, \ldots, 0)}$.

Lemma 2: The Weyl denominator formula for $\mathfrak{sl}_n(\mathbb{C})$ becomes

Vandermonde determinant formula
$$\prod_{i<j} (x_i - x_j) = \sum_{\sigma \in S(n)} \mathrm{sgn}(\sigma)\, x_{\sigma(1)}^{n-1} x_{\sigma(2)}^{n-2} \cdots x_{\sigma(n)}^{0} \quad \text{with } \mathrm{sgn}(\sigma) = (-1)^{\ell(\sigma)}$$

As $\rho = (n-1, n-2, \ldots, 1, 0)$, we get:

$$\Delta = \prod_{\alpha \in R_+} (e^{\alpha/2} - e^{-\alpha/2}) = e^\rho \prod_{\alpha \in R_+} (1 - e^{-\alpha}) = x_1^{n-1} x_2^{n-2} \cdots x_n^0 \prod_{i<j} \left(1 - \frac{x_j}{x_i}\right) = \prod_{i<j} (x_i - x_j)$$

$\|$ Weyl denominator f-la

$$\sum_{\sigma \in S(n)} (-1)^{\ell(\sigma)} e^{\sigma(\rho)} = \sum_{\sigma \in S(n)} \mathrm{sgn}(\sigma)\, x_{\sigma(1)}^{n-1} x_{\sigma(2)}^{n-2} \cdots x_{\sigma(n)}^{0}$$

Lecture #27

Let us finally see what the Weyl character formula looks like in the $\mathfrak{sl}_n$-case.

Theorem 3: For $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n_{\geq 0}$ with $\lambda_1 \geq \ldots \geq \lambda_n$, the character of the corresponding $\mathfrak{sl}_n$ irreducible module L_λ is given by:

$$\chi_{L_\lambda} = \frac{\det\left(x_i^{\lambda_j + n - j}\right)_{1 \leq i,j \leq n}}{\det\left(x_i^{n-j}\right)} \leftarrow \text{Schur function } S_\lambda(x_1, \ldots, x_n)$$

Remark: By Proposition 1, we see that Schur functions form a basis of S_n-symmetric polynomials in n variables. This is a very important basis of this ring!