Lecture #28

Last time: ended with the discussion of sl_n-representations.

Today: Representations of $GL_n(\mathbb{C})$

Recall: Any fin. dim. G-module is also a $Lie(G)$-module, while for the inverse implication we need G to be simply connected.

[Exercise: Show that $SL_n(\mathbb{C})$ are simply connected

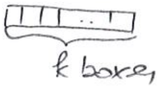
Hint: You may wish to relate $SL_n(\mathbb{C})$ to $SU(n)$ and use [Homework 3, Problem 1]

Thus, finite dimensional $SL_n(\mathbb{C})$-modules are completely reducible, and the irreducible ones are $\{L_\lambda \mid \lambda=(\lambda_1,\lambda_2,\dots,\lambda_{n-1},0)\in\mathbb{Z}^n \text{ with } \lambda_1\geq\lambda_2\geq\dots\geq\lambda_{n-1}\geq 0\}$

Proposition 1: ↗

Recall Example 4 from Lecture 27:

- $S^k\mathbb{C}^n$ is an irreducible highest weight sl_n-module with the highest vector $e_1^{\otimes k}$, where $\{e_1,\dots,e_n\}$ is the standard basis of $\mathbb{C}^n$, and its weight is $(k,0,\dots,0)=k\cdot\underset{1^{st}\text{ fundamental weight}}{\underline{\omega_1}}$ which we depict by



- $\Lambda^k\mathbb{C}^n$ $(1\leq k\leq n)$ is irreducible with $e_1\wedge e_2\wedge\dots\wedge e_k$ being the highest vector, and its weight is $(\underbrace{1,\dots,1}_{k},0,\dots,0)=\underset{k^{th}\text{ fund. weight}}{\underline{\omega_k}}$ which we depict by

Therefore: Any $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_{n-1},0)\in\mathbb{Z}^n$ with $\lambda_1\geq\lambda_2\geq\dots\geq\lambda_{n-1}\geq 0$ can be written as

$$\underset{\Downarrow}{\lambda}=(\lambda_1-\lambda_2)\cdot(1,0,0,0,\dots)+(\lambda_2-\lambda_3)(1,1,0,0,\dots)+(\lambda_3-\lambda_4)(1,1,1,0,\dots)$$

$$\boxed{\lambda=(\lambda_1-\lambda_2)\omega_1+(\lambda_2-\lambda_3)\omega_2+\dots+(\lambda_{n-1}-\underset{\overset{\shortparallel}{0}}{\lambda_n})\omega_{n-1}}$$

so λ as above correspond to $\{m_1\omega_1+\dots+m_{n-1}\omega_{n-1} \mid m_k\in\mathbb{Z}_{\geq 0}\}$ with $m_k=\lambda_k-\lambda_{k+1}$

Moreover, as we recalled above each fundamental representation L_{ω_k} is just $\Lambda^k\mathbb{C}^n$, i.e. the exterior power of the tautological repr-n $\mathbb{C}^n$

In particular, we note $\Lambda^n\mathbb{C}^n$ is the trivial repr-n of $sl_n(\mathbb{C})$ and $SL_n(\mathbb{C})$

(Exercise (easy): Show $\Lambda^k V^*\simeq\Lambda^{n-k}V$ $\forall 1\leq k\leq n$

We have the following general result:

Proposition 2: Let $\mathfrak{g}$ be a s.s. Lie algebra, and $\lambda = \sum m_i \omega_i \in P_+$ – dominant integral weight. Then L_λ can be realized as $\mathfrak{g}$-submodule of $\bigotimes_i L_{\omega_i}^{\otimes m_i}$ generated by $v = \bigotimes_i v_{\omega_i}^{\otimes m_i}$ (here, v_{ω_i} is the highest weight vector in L_{ω_i})

▶ Let $V := \mathcal{U}(\mathfrak{g}) v \subseteq \bigotimes L_{\omega_i}^{\otimes m_i}$ be the above $\mathfrak{g}$-submodule. Then it's completely reducible, and v is a highest weight vector $\Rightarrow V \simeq L_\lambda \oplus \bigoplus_k L_{\nu_k}$. But similarly to the proof of [Lecture 27, Thm 1 b)] we immediately deduce there are in fact no L_{ν_k}'s (as v generates both V and L_λ). $\blacksquare$

In our context from p. 1, we thus obtain:

Corollary 1: For $\lambda = \sum_{k=0}^{n-1} m_k \omega_k = (m_1 + \ldots + m_{n-1}, m_2 + m_3 + \_ + m_{n-1}, \to m_{n-1}, 0)$, the irreducible $SL_n(\mathbb{C})$-module L_λ is generated inside $\bigotimes_{k=1}^{n-1} (\Lambda^k \mathbb{C}^n)^{\otimes m_k}$ by the tensor product $\bigotimes_{k=1}^{n-1} (e_1 \wedge \ldots \wedge e_k)^{\otimes m_k}$. In particular, all irreducible fin.dim representations of $SL_n(\mathbb{C})$ can be always realized inside $(\mathbb{C}^n)^{\otimes(\sum k m_k)} = (\mathbb{C}^n)^{|\lambda|}$

Let us now move to representations of $GL_n(\mathbb{C})$. Note that $\mathbb{C}^* = \{\lambda \cdot \mathrm{Id} \mid \lambda \neq 0\}$ is a subgp of $GL_n(\mathbb{C})$ and any $A \in GL_n(\mathbb{C})$ can be written as $A = \lambda \cdot B$, $\lambda \in \mathbb{C}^*$, $B \in SL_n(\mathbb{C})$. However, this decomposition is not unique as:

$$\mathbb{C}^* \cap SL_n(\mathbb{C}) = \{\lambda \cdot \mathrm{Id} \mid \lambda^n = 1\} = \underbrace{\mu_n}_{\text{gp of roots of 1 of order } n} \simeq \mathbb{Z}/_{n\mathbb{Z}}$$

Therefore, we have the following short exact sequence:

$$\begin{tikzcd} 1 \arrow[r] & \mu_n \arrow[r] & \mathbb{C}^* \times SL_n(\mathbb{C}) \arrow[r] & GL_n(\mathbb{C}) \arrow[r] & 1 \\ & z \arrow[r, mapsto] & (z^{-1}, z \cdot \mathrm{Id}) & & \end{tikzcd}$$

Hence, any representation of $GL_n(\mathbb{C})$ is a repr-n of $\mathbb{C}^* \times SL_n(\mathbb{C})$, and any repr. of $\mathbb{C}^* \times SL_n(\mathbb{C})$ on which μ_n acts trivially descends to a repr. of $GL_n(\mathbb{C})$

To this end, we shall first consider repr-s of $\mathbb{C}^*$. Note that $\mathrm{Lie}(\mathbb{C}^*)=\mathbb{C}$ is an abelian 1dim Lie algebra, hence its representation is just a pair (V,α) where V- vect. space, $\alpha\in\mathrm{End}(V)$ is the image of $1\in\mathbb{C}$. On the other hand, the exponential map $\begin{array}{ccc}\mathbb{C} & \longrightarrow & \mathbb{C}^* \\ \rotatebox{90}{$\in$} & & \\ x & \longmapsto & e^{2\pi i x}\end{array}$ is surjective with the kernel $\mathbb{Z}$.

Therefore, a pair (V,α) with fin.dim. V exponentiates to a representation of $\mathbb{C}^*$ if and only if $\boxed{e^{2\pi i\alpha}=\mathrm{Id}_V}$

(Exercise (easy): $e^{2\pi i\alpha}=\mathrm{Id}\iff\alpha$ is diagonalizable with $\mathbb{Z}$ eigenvalues

Corollary 2: Finite dimensional repr-s of $\mathbb{C}^*$ are completely reducible with irreducible $\chi_r:\begin{array}{ccc}\mathbb{C}^* & \longrightarrow & \mathrm{End}(\mathbb{C}) \\ \rotatebox{90}{$\in$} & & \\ z & \longmapsto & z^r\end{array}$ being 1dim, labelled by $r\in\mathbb{Z}$

Combining the above discussion, we thus obtain:

Proposition 3: Irreducible finite dimensional representations of $\mathbb{C}^*\times SL_n(\mathbb{C})$ are

$\chi_r\otimes L_\lambda$ with $r\in\mathbb{Z}$, λ as above

Moreover, those that factor through $GL_n(\mathbb{C})$ satisfy $r-\sum_{i=1}^{n-1}\lambda_i\in n\mathbb{Z}$

(Exercise: fill in remaining details in the proof.

If we write λ as $\lambda=(m_1+\ldots+m_{n-1}, m_2+\ldots+m_{n-1},\ldots, m_{n-1}, 0)$ and define $m_n:=\frac{r-\sum\lambda_i}{n}$ (which is integer by Prop 3) we can easily see that the highest weight of above $\chi_r\otimes L_\lambda$ equals $(m_1+\ldots+m_{n-1}+m_n, m_2+\ldots+m_{n-1}+m_n,\ldots, m_{n-1}+m_n, m_n)$.

Corollary 3: Highest weights of irred. fin. dim. $GL_n(\mathbb{C})$-modules are

$$\{\lambda=(\lambda_1,\ldots,\lambda_n)\in\mathbb{Z}^n \mid \lambda_1\geq\lambda_2\geq\ldots\geq\lambda_n\}$$

In other words they have the form:

$$\boxed{\{\lambda=m_1\omega_1+\ldots+m_{n-1}\omega_{n-1}+m_n\omega_n \mid m_1,\ldots,m_{n-1}\in\mathbb{Z}_{\geq0},\ m_n\in\mathbb{Z}\}} \leftarrow \begin{array}{l}\omega_k\ (k<n) \text{ as above}\\ \omega_n=(1,\ldots,1)\end{array}$$

Lecture #28

Let us note that while $\Lambda^n\mathbb{C}^n$ was a trivial $SL_n(\mathbb{C})$-module, as a $GL_n(\mathbb{C})$-module it's precisely L_{ω_n}. Moreover, its dual $(\Lambda^n\mathbb{C}^n)^*$ is also irreducible, namely $L_{-\omega_n}$. Thus, generalizing Corollary 1, we get:

Corollary 4: For $\lambda = m_1\omega_1+\_\_+m_{n-1}\omega_{n-1}+m_n\omega_n$ ($m_1,\_, m_{n-1}\in\mathbb{Z}_{\geq 0}$, $m_n\in\mathbb{Z}$), the $GL_n(\mathbb{C})$-module L_λ can be realized inside $\bigotimes\limits_{k=1}^{n}(\Lambda^k\mathbb{C}^n)^{\otimes m_k}$.

↑ if $m_n<0$, then $(\Lambda^n\mathbb{C}^n)^{\otimes m_n} = ((\Lambda^n\mathbb{C}^n)^*)^{\otimes(-m_n)}$

Def 1: Representations L_λ with $\lambda=\sum\limits_{k=1}^{n} m_k\omega_k$ and all $m_k\in\mathbb{Z}_{\geq 0}$ are called *polynomial*.

The terminology depicts the fact that all matrix coefficients of $\rho: GL_n(\mathbb{C})\longrightarrow End(L_\lambda)$ are polynomials in matrix entries of $A\in GL_n(\mathbb{C})$, and thus can be continuously extended to a representation of a semigroup $Mat_{n\times n}(\mathbb{C})$.

(Exercise: $m_n\geq 0 \iff L_\lambda$ occurs inside $V^{\otimes N}$ for some N.

Thus: irreducible polynomial finite dimensional representations of $GL_n(\mathbb{C})$ are parametrized by

$$\boxed{\{\lambda=(\lambda_1,\_\_,\lambda_n)\in\mathbb{Z}^n_{\geq 0} \mid \lambda_1\geq\lambda_2\geq\_\_\geq\lambda_n\}}$$

Moreover, L_λ occurs inside $(\mathbb{C}^n)^{\otimes|\lambda|}$, where $|\lambda|=\lambda_1+\ldots+\lambda_n$ = # boxes in corresponding Young diagram.

On the other hand, we know that any f.dim. $GL_n(\mathbb{C})$-module is completely reducible. We can thus, decompose $(\mathbb{C}^n)^{\otimes N}$ into irreducible $GL_n(\mathbb{C})$-mod:

$\forall N\geq 1$:
$$(\mathbb{C}^n)^{\otimes N} = \bigoplus_{\substack{|\lambda|=N \\ \lambda=(\lambda_1,\_,\lambda_n)\in\mathbb{Z}^n_{\geq 0} \\ \lambda_1\geq\ldots\geq\lambda_n}} L_\lambda\otimes U_\lambda, \quad \text{where } U_\lambda := Hom_{GL_n(\mathbb{C})}(L_\lambda, (\mathbb{C}^n)^{\otimes N})$$
are just vector spaces for now.

Equivalently, we can say that the above summation is over all size N Young diagrams of length $\leq n$ (i.e. at most n nonzero rows). We shall now equip U_λ with an additional structure of S_N-modules, where S_N denotes the symmetric group in $\{1,\_\, N\}$.

Lecture #28

Consider an action $S_N \curvearrowright (\mathbb{C}^n)^{\otimes N}$ via

$$\sigma.(v_1 \otimes \dots \otimes v_N) = v_{\sigma^{-1}(1)} \otimes v_{\sigma^{-1}(2)} \otimes \dots \otimes v_{\sigma^{-1}(N)}$$

This action obviously commutes with the diagonal action $GL_n(\mathbb{C}) \curvearrowright (\mathbb{C}^n)^{\otimes}$

$$g.(v_1 \otimes \dots \otimes v_N) = (gv_1) \otimes (gv_2) \otimes \dots \otimes (gv_N).$$

We shall depict this by

$$\boxed{GL_n(\mathbb{C}) \curvearrowright (\mathbb{C}^n)^{\otimes N} \circlearrowleft S_N}$$

As they commute, we obtain a natural action of S_N on each multiplicity space

$$\boxed{S_N \curvearrowright U_\lambda}$$

Def 2: Let $A := \mathrm{Im}\,(U(\mathfrak{gl}_n(\mathbb{C})) \longrightarrow \mathrm{End}((\mathbb{C}^n)^{\otimes N}))$ – "Schur algebra"

$B := \mathrm{Im}\,(\mathbb{C}[S_N] \longrightarrow \mathrm{End}((\mathbb{C}^n)^{\otimes N}))$ – "centralizer algebra"

Theorem 1 (Schur-Weyl duality):

a) The centralizer of A is B and the centralizer of B is A.

b) If a partition λ of N has at most n parts, then the representation U_λ of B is irreducible, nonzero, and pairwise non-isomorphic all irreducible representations of B.

c) If $n \geq N$, then $\{U_\lambda\}_{|\lambda|=N}$ exhaust all irreducible representations of S_N.

Note that the centralizer of B is $Z_B = S^N(\mathrm{End}(\mathbb{C}^{\otimes n}))$. We shall thus start with two general results on symmetric powers $S^N U$.

Lemma 1: For any $\mathbb{C}$-vector space U, $S^N U$ is spanned by $\{x^{\otimes N} = x \otimes \dots \otimes x \mid x \in U\}$

For example, if $N=2$ this follows from $x \otimes y + y \otimes x = \frac{1}{2}\left((x+y)^{\otimes 2} - x^{\otimes 2} - y^{\otimes 2}\right)$

By [Hwk 12, Problem 6] = [Hwk 4, Problem 6], $S^N U$ is an irreducible $GL(U)$-module. But span $\{x^{\otimes N} \mid x \in U\}$ is clearly a nonzero $GL(U)$-submodule. Hence, it equals all $S^N U$

Lecture #28

While the above lemma provided a set of elements that linearly span $S^N U$, we wish to refine it to a set of elements that generate $S^N U$ as algebra (given U is an algebra itself)

Lemma 2: If U is an associative $\mathbb{C}$-algebra, then $S^N U$ is generated by

$$\Delta^{(N)}(x) := x\otimes 1\otimes \ldots \otimes 1 + 1\otimes x\otimes \ldots \otimes 1 + \ldots + 1\otimes \ldots \otimes 1\otimes x, \ x\in U$$

Let's illustrate it for $N=2$: $x\otimes x = \frac{1}{2}\left((x\otimes 1+1\otimes x)^2 - (x^2\otimes 1+1\otimes x^2)\right) = \frac{\Delta^{(2)}(x)^2 - \Delta^{(2)}(x^2)}{2}$

▶ Due to Lemma 1, it suffices to express $x^{\otimes N}$ via $\Delta^{(N)}(?)$. This follows from the following explicit formula:

$$x^{\otimes N} = p(\Delta^{(N)}(x), \Delta^{(N)}(x^2), \longrightarrow \Delta^{(N)}(x^N))$$

where p is a polynomial in N variables satisfying

$$y_1 \cdots y_N = p\left(\sum y_i, \sum y_i^2, \longrightarrow \sum y_i^N\right)$$

We leave details as a simple exercise ■

Combining Lemma 2 with $Z_B = S^N(\mathrm{End}(\mathbb{C}^n))$ and the observation that $\Delta^{(N)}(x)$ for $x\in \mathrm{End}(\mathbb{C}^n)$ is precisely how x acts on $(\mathbb{C}^n)^{\otimes N}$, we get:

$$\boxed{A = \text{centralizer of } B}$$

proving one part of Thm 1a). In fact, all the rest follows from the general algebraic statement:

Proposition 4 (The Double Centralizer Lemma): Let V be a fin. dim. vector space, $A, B \subseteq \mathrm{End}(V)$ be subalgebras, s.t. B is isomorphic to a direct sum of matrix algebras and A = centralizer of B. Then: A is also isomorphic to a direct sum of matrix algebras, $V = \bigoplus_{i=1}^{m} W_i \otimes U_i$ where W_i runs through all irred. A-modules, U_i - irred. B-modules, and B = centralizer of A. In particular, we get bijection b/w {irred. A-mod} $\overset{1:1}{\longleftrightarrow}$ {irred. B-mod}

Remark: An algebra A is isomorphic to a direct sum of matrix algebras iff A is semisimple (i.e. no nonzero elt of A acts trivially on all irreducible A-modules) or equivalently if any f.d. A-module is completely reducible

► The result follows immediately from decomposing V as B-module

$$V = \bigoplus_i W_i \otimes U_i \quad \text{with} \quad W_i := \mathrm{Hom}_B(U_i, V)$$

as then $A = \oplus\, \mathrm{End}(W_i)$, $B = \oplus\, \mathrm{End}(U_i)$ ∎

In our case, B is the quotient of $\mathbb{C}[S_N]$. As every f.d. S_N-module is completely reducible, the same holds for B-modules, and hence by above Rmk the algebra B is isomorphic to a direct sum of matrix algebras.

Hence: Prop 4 applies and we get:

1) B = centralizer of A, finishing the proof of Thm 1a)

2) $\{U_\lambda\}_{\substack{\mathrm{length}(\lambda)\le n \\ |\lambda| = N}}$ are pairwise non-isomorphic and provide all irreducible B-modules, thus establishing Thm 1b).

Finally, to prove Thm 1c) it suffices to show $B \simeq \mathbb{C}[S_N]$, i.e. the map $\mathbb{C}[S_N] \longrightarrow \mathrm{End}((\mathbb{C}^n)^{\otimes N})$ is injective for $n \geq N$. To this end, we note that $\{\sigma_\circ(e_1 \otimes e_2 \otimes \ldots \otimes e_N) \mid \sigma \in S_N\}$ are clearly lin. indep. decomposable tensors.

This completes our proof of the Schur-Weyl duality! ∎

Remark: We also have $A = \mathrm{span}\{g^{\otimes N} \mid g \in \mathrm{End}(\mathbb{C}^n)\}$, i.e.

$$\mathrm{Im}\left(U(\mathfrak{gl}_n(\mathbb{C})) \longrightarrow \mathrm{End}((\mathbb{C}^n)^{\otimes N})\right) = \mathrm{span}\left(\mathrm{Im}\left(GL_n(\mathbb{C}) \longrightarrow \mathrm{End}((\mathbb{C}^n)^{\otimes N})\right)\right)$$