Lecture #29

- Last time: Schur–Weyl duality

$$\boxed{\begin{matrix} GL(V) \curvearrowright V^{\otimes N} \curvearrowleft S_N \\ gl(V) \end{matrix}}$$

← double centralizing images
$A = \mathrm{Im}(U(gl(V)) \to \mathrm{End}(V^{\otimes N})) =$ span of $\mathrm{Im}(GL(V) \to GL(V^{\otimes N}))$
$B = \mathrm{Im}(\mathbb{C}[S_N] \to \mathrm{End}(V^{\otimes N}))$

1) $$\boxed{V^{\otimes N} \simeq \bigoplus_{|\lambda|=N} L_\lambda \otimes U_\lambda}$$
λ – Young diagram of length $\le \dim V$
isomorphism of $A\otimes B$-modules

and we get bijection

2) $\{$irreducible polynomial $GL(V)$-representations$\} \xleftrightarrow{1:1} \{$irreducible B-repr-s$\}$
$L_\lambda \longleftrightarrow U_\lambda$

3) If $\underbrace{\dim V}_{=:n} \ge N \Rightarrow B = \mathbb{C}[S_N] \Rightarrow \boxed{\{U_\lambda\}_{|\lambda|=N} = \text{all irreducible } S_N\text{-modules}}$

- Let's see what the above theorem implies on the level of characters.

Recall (from Lecture 27): Character of L_λ, viewed as usual character of the group $GL(V) \simeq GL_n(\mathbb{C})$ module L_λ is given by Schur polynomial

$$\boxed{S_\lambda(x_1, \dots, x_n) = \frac{\det(x_i^{\lambda_j+N-j})}{\det(x_i^{N-j})}}$$

$x_1, \dots, x_n$ – eigenvalues of $g \in GL_n(\mathbb{C})$

Pick a diagonal element $x = \begin{pmatrix} x_1 & & 0 \\ & x_2 & \\ 0 & & \ddots\, x_n \end{pmatrix} \in GL_n(\mathbb{C})$. Also pick any permutation $\sigma \in S_N$, and let it have m_k cycles of length k. We shall now compute the trace of $x\otimes\sigma$-action on $V^{\otimes N}$ in two different ways:

(i) just from def-n: $tr_{V^{\otimes N}}(x\otimes\sigma) = \prod_k (x_1^k + \dots + x_n^k)^{m_k}$ ← explain!

(ii) from Schur–Weyl: $tr_{V^{\otimes N}}(x\otimes\sigma) = \sum_\lambda S_\lambda(x_1, \dots, x_n)\cdot \chi_{U_\lambda}(\sigma)$

$$\Rightarrow \boxed{\sum_\lambda \chi_{U_\lambda}(\sigma)\cdot \det(x_i^{\lambda_j+N-j}) = \det(x_i^{N-j})\cdot \prod_k (x_1^k + \dots + x_n^k)^{m_k} = \prod_{i<j}(x_i - x_j)\prod_k (x_1^k + \dots + x_n^k)^{m_k}}$$

Note that if we order monomials lexicographically by first comparing power of x_1, then x_2, etc, then the max monomial of $\det(x_i^{\lambda_j+N-j})$ is $x_1^{\lambda_1+N-1} \cdots x_N^{\lambda_N}$.

The above implies the well-known Frobenius character formula:

Theorem 1 (Frobenius character formula): The character value $\chi_{U_\lambda}(\sigma)$ equals the coefficient of $x_1^{\lambda_1+N-1}\cdots x_N^{\lambda_N}$ in

$$\prod_{1\le i<j\le n}(x_i-x_j)\cdot\prod_{k\ge 1}(x_1^k+\ldots+x_n^k)^{m_k}$$

• Another beautiful application of the Schur-Weyl duality is the following result. Let V,W be two finite-dimensional $\mathbb{C}$ vector spaces.

Theorem 2 (Howe duality): The symmetric power $S^n(V\otimes W)$ decomposes as

$$S^n(V\otimes W)\simeq\bigoplus_{\substack{\lambda\text{-Young diagram}\\ \text{of size }|\lambda|=n}} S^\lambda V\otimes S^\lambda W$$

as $GL(V)\times GL(W)$-modules, where $S^\lambda V=\mathrm{Hom}_{\mathbb{C}[S_n]}(U_\lambda, V^{\otimes n})=\begin{cases}L_\lambda, & \text{if length}(\lambda)\le\dim V\\ 0, & \text{otherwise}\end{cases}$

▶ By definition: $S^n(V\otimes W)=[(V\otimes W)^{\otimes n}]^{S_n}\simeq[V^{\otimes n}\otimes W^{\otimes n}]^{S_n}$ ← here, S_n diagonally acts on $V^{\otimes n}$, $W^{\otimes n}$

Schur-Weyl $\Rightarrow$ $\left.\begin{aligned} V^{\otimes n}&=\bigoplus_{|\lambda|=n}S^\lambda V\otimes U_\lambda\\ W^{\otimes n}&=\bigoplus_{|\lambda|=n}S^\lambda W\otimes U_\lambda\end{aligned}\right\}\Rightarrow \boxed{S^n(V\otimes W)\simeq\bigoplus_{\substack{|\lambda|=n\\|\mu|=n}}S^\lambda V\otimes S^\mu W\otimes(U_\lambda\otimes U_\mu)^{S_n}}$

But: $U_\lambda\simeq U_\lambda^*$ $\forall\lambda$, since $\chi_{U_\lambda^*}=\overline{\chi_{U_\lambda}}=\chi_{U_\lambda}$ as χ_{U_λ} takes $\mathbb{Z}$-values by Thm 1

$\Rightarrow (U_\lambda\otimes U_\mu)^{S_n}\simeq(U_\lambda^*\otimes U_\mu)^{S_n}\simeq\mathrm{Hom}_{S_n}(U_\lambda,U_\mu)\simeq\begin{cases}\mathbb{C}, & \text{if }\lambda=\mu\\ 0, & \text{otherwise}\end{cases}$ ∎

[Rmk: The above definition of $S^\lambda V$ is such that Schur-Weyl duality becomes:
$V^{\otimes N}=\bigoplus_\lambda S^\lambda V\otimes U_\lambda$ as $\mathfrak{gl}(V)\times S_N$-module (i.e. we lifted "length$(\lambda)\le\dim V$")

As an immediate corollary of Theorem 2, we get:

Corollary 1 (Cauchy identity): Consider two tuples of variables: $(x_1,\ldots,x_k)$, $(y_1,\ldots,y_\ell)$

Then:
$$\prod_{\substack{1\le i\le k\\1\le j\le \ell}}\frac{1}{1-zx_iy_j}=\sum_\lambda S_\lambda(x_1,\ldots,x_k)S_\lambda(y_1,\ldots,y_\ell)z^{|\lambda|}$$

▶ Compute trace of the action of $x\otimes y$ with $x=\begin{pmatrix}x_1&&0\\&\ddots&\\0&&x_k\end{pmatrix}\in GL(\mathbb{C}^k)$, $y=\begin{pmatrix}y_1&&0\\&\ddots&\\0&&y_\ell\end{pmatrix}\in GL(\mathbb{C}^\ell)$ and use the Molien formula $\sum_{m\ge0}\mathrm{tr}(S^m\alpha)z^m=\frac{1}{\det(1-z\cdot\alpha)}$ for any linear $\alpha: U\to U$ ∎

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• Levi decomposition

The goal for the rest of today is to prove the Levi theorem, which plays the fundamental role in the proofs of Ado's thm and 3d Fundamental thm of Lie theory. Let's recall the formulation – see Lecture #11.

Thm (Levi thm): Any finite-dimensional Lie algebra $\mathfrak{g}$ (over a field $\mathbb{k}$ of $\mathrm{char}(\mathbb{k})=0$) can be written as a direct sum $\mathfrak{g} \overset{\text{v. space}}{=\!=} \underbrace{\mathrm{rad}(\mathfrak{g})}_{\text{ideal}} \oplus \underbrace{\mathfrak{g}_{ss}}_{\text{Lie subalgebra}}$ with $\mathfrak{g}_{ss}$ - semisimple

Rmk: a) Recall that by [Lecture 11, Lemma 2], the quotient $\mathfrak{g}/\mathrm{rad}(\mathfrak{g})$ is semisimple and by above is isomorphic to $\mathfrak{g}_{ss}$, so that as Lie algebras $\mathfrak{g}_{ss}$ are choice independent. However, there may be different embeddings $\mathfrak{g}/\mathrm{rad}(\mathfrak{g}) \hookrightarrow \mathfrak{g}$ as above

b) As noted in the Remark after Levi thm in Lecture 11, the above implies
$$\mathfrak{g} \simeq \mathfrak{g}_{ss} \ltimes \mathrm{rad}(\mathfrak{g})$$

Proof of thm

To shorten our notations, we shall use $\mathfrak{r}$ for $\mathrm{rad}(\mathfrak{g})$, while $\mathfrak{a}$ for $\mathfrak{g}_{s.s.}$

Case 1: $\mathfrak{r} = Z(\mathfrak{g})$, i.e. $\mathfrak{g}$ - reductive

This case was already treated in [Lecture 13-14, Corollary 6]

Namely the adjoint action $\mathfrak{g} \overset{ad}{\curvearrowright} \mathfrak{g}$ factorizes through $\mathfrak{g}/\mathfrak{r} \overset{ad}{\curvearrowright} \mathfrak{g}$, and $\mathfrak{r} \subseteq \mathfrak{g}$ is a subrepresentation. However, as $\mathfrak{g}/\mathfrak{r}$ is semisimple, we know that any fin. dim. module is completely reducible [Lecture 13-14, Thm 2], hence: $\mathfrak{g} \simeq \mathfrak{r} \oplus \mathfrak{a}$ as $\mathfrak{g}/\mathfrak{r}$-modules. In particular, $\mathfrak{a} \simeq \mathfrak{g}/\mathfrak{r}$ - semisimple and actually $\mathfrak{a} \subseteq \mathfrak{g}$ is not just a subalg. but an ideal. ✓

Case 2: $\begin{cases} \mathfrak{r} \neq Z(\mathfrak{g}) \iff [\mathfrak{g}, \mathfrak{r}] \neq 0 \\ \mathfrak{r} \text{ contains no nonzero proper ideals} \end{cases} \Rightarrow \begin{cases} [\mathfrak{g},\mathfrak{r}] = \mathfrak{r} \text{ as otherwise } [\mathfrak{g},\mathfrak{r}] \text{ would be an ideal in } \mathfrak{r} \\ [\mathfrak{r},\mathfrak{r}] = 0 \text{ as } [\mathfrak{r},\mathfrak{r}] \subsetneq \mathfrak{r} \text{ and is an ideal in } \mathfrak{r} \end{cases}$

In this case, instead of considering adjoint action, we consider the following:

$$\boxed{\mathfrak{g} \overset{\varkappa}{\curvearrowright} \mathfrak{gl}(\mathfrak{g}), \text{ i.e. } \varkappa: \mathfrak{g} \to \mathrm{End}(\mathfrak{gl}(\mathfrak{g})), \text{ given by } \varkappa(x)\varphi = [\mathrm{ad}(x), \varphi] \quad \forall x \in \mathfrak{g},\ \varphi \in \mathrm{End}(V)}$$

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We shall now define the following 3 $\mathfrak{g}$-invariant subspaces of $gl(\mathfrak{g})$ under $\varkappa$:

$$\boxed{\begin{aligned} &\mathcal{A} := \{\varphi \in gl(\mathfrak{g}) \mid \varphi(\mathfrak{g}) \subseteq \mathfrak{r},\ \varphi|_{\mathfrak{r}} = c \cdot Id_{\mathfrak{r}} \text{ for some } c \in \mathbb{k}\} \\ &\mathcal{B} := \{\varphi \in \mathcal{A} \mid \varphi|_{\mathfrak{r}} = 0\} \\ &\mathcal{C} := \{ad(x) \mid x \in \mathfrak{r}\} \end{aligned}} \qquad \mathcal{A} \supseteq \mathcal{B} \supseteq \mathcal{C}$$

Properties:

1) $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are indeed $\varkappa(\mathfrak{g})$-invariant

- For $\varphi \in \mathcal{A}$, $x \in \mathfrak{g}$:

$$\left.\begin{aligned} &(ad(x)\varphi - \varphi\, ad(x))(y) = [x, \varphi(y)] - \varphi([x,y]) \subseteq \mathfrak{r} \quad \forall y \in \mathfrak{g} \\ &\text{if } y \in \mathfrak{r} \Rightarrow \varphi(y) = cy \Rightarrow (ad(x)\varphi - \varphi\, ad(x))(y) = 0 \end{aligned}\right\} \Rightarrow \varkappa(x)\varphi \in \mathcal{B} \subseteq \mathcal{A} \quad \forall x \in \mathfrak{g},\ \varphi \in \mathcal{A}$$

- For $\varphi \in \mathcal{B}$ – same check
- For $\mathcal{C}$: $[ad(x), ad(y)] = ad[x,y]$ ▫

2) $\dim(\mathcal{A}/\mathcal{B}) = 1$

3) $\varkappa(\mathfrak{g})\mathcal{A} \subseteq \mathcal{B}$ ← checked above

4) $\varkappa(\mathfrak{r})\mathcal{A} \subseteq \mathcal{C}$

$$\forall x \in \mathfrak{r}, \varphi \in \mathcal{A}, y \in \mathfrak{g}: \ (\varkappa(x)\varphi)(y) = [\underset{\mathfrak{r}}{x}, \underset{\mathfrak{r}}{\varphi(y)}] - \varphi([\underset{\mathfrak{r}}{x}, \underset{\mathfrak{g}}{y}]) = -\varphi([x,y]) = -c[x,y] = ad(-cx)y$$ ▫

Let's now consider the corresponding quotient representations $\mathcal{A}/\mathcal{B}$, $\mathcal{A}/\mathcal{C}$ of $\mathfrak{g}/\mathfrak{r}$.
We have a natural projection

$$\boxed{\pi: \mathcal{A}/\mathcal{C} \twoheadrightarrow \underbrace{\mathcal{A}/\mathcal{B}}_{1\text{dim } \mathfrak{g}/\mathfrak{r}\text{-repr.}}} \rightsquigarrow \ \text{Ker}(\pi) \subsetneq \mathcal{A}/\mathcal{C} \text{ is a codim 1 } \mathfrak{g}/\mathfrak{r}\text{-submodule}$$

But $\mathfrak{g}/\mathfrak{r}$ - semisimple $\Rightarrow$ any f.dim. module is completely reducible, hence,

$$\boxed{\mathcal{A}/\mathcal{C} \simeq \text{Ker}(\pi) \oplus \underbrace{\mathbb{k}(\varphi_0 + \mathcal{C})}_{1\text{dim complement to Ker}(\pi)} \text{ as } \mathfrak{g}/\mathfrak{r}\text{-modules}}.$$

Rescaling φ_0 we can assume $\varphi_0 \in \mathcal{A}$ is s.t. $\varphi_0|_{\mathfrak{r}} = Id_{\mathfrak{r}}$.

Exercise (easy): Any 1-dim repr. of a semisimple Lie algebra is trivial!

Hence, $\varkappa(\mathfrak{g})\varphi_0 \subseteq \mathcal{C}$. Here is our choice of $\mathfrak{a}$ (clearly a subalgebra of $\mathfrak{g}$):

$$\boxed{\mathfrak{a} := \{x \in \mathfrak{g} \mid \varkappa(x)\varphi_0 = 0\}}$$

We shall now check $\mathfrak{a}$ is as needed.

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(Continuation of Step 2)

Properties of $\mathfrak{a}$:

1) $\mathfrak{a} \cap \mathfrak{r} = 0$

▸ If $x \in \mathfrak{r} \Rightarrow \varkappa(x)\varphi_0 = \mathrm{ad}(-x)$
If $x \in \mathfrak{a} \Rightarrow \varkappa(x)\varphi_0 = 0$
$\Big\} \Rightarrow \mathrm{ad}(x) = 0$ for $x \in \mathfrak{a} \cap \mathfrak{r}$.

Thus, if $x \neq 0$, then $\mathbb{k}x \subsetneq \mathfrak{r}$ is a nonzero proper ideal, contradicting our assumptions ▥

2) $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{r}$

▸ Pick any $x \in \mathfrak{g}$. By above: $\varkappa(x)\varphi_0 \in \mathcal{B} \Rightarrow \exists y \in \mathfrak{r}$ s.t. $\varkappa(x)\varphi_0 = \mathrm{ad}(y)$
But $\varkappa(y)\varphi_0 = \mathrm{ad}(-y)$
$\Big\} \Rightarrow \varkappa(x+y)\varphi_0 = 0.$

Thus: $x = (x+y) + (-y)$ with $x+y \in \mathfrak{a}$, $-y \in \mathfrak{r}$

The uniqueness follows from 1) above ▥

This establishes the theorem in Case 2. ✓

Case 3: general case
(not to get into Cases 1-2, can assume $\dim(\mathfrak{r}) > 1$ and there exists a proper nonzero ideal $0 \subsetneq \mathfrak{r}' \subsetneq \mathfrak{r} \subseteq \mathfrak{g}$)

We shall argue by induction on $\dim(\mathfrak{g})$.

Exercise (easy): $\mathrm{rad}(\mathfrak{g}/\mathfrak{r}') = \mathfrak{r}/\mathfrak{r}'$.

As $\dim(\mathfrak{g}/\mathfrak{r}') < \dim(\mathfrak{g})$, the induction hypothesis implies $\boxed{\mathfrak{g}/\mathfrak{r}' = \mathfrak{r}/\mathfrak{r}' \oplus \underbrace{\mathfrak{a}'}_{\text{subalg. in } \mathfrak{g}/\mathfrak{r}'}}$

To get back to $\mathfrak{g}$, consider the natural quotient map $\varpi: \mathfrak{g} \twoheadrightarrow \mathfrak{g}/\mathfrak{r}'$ and let's look at $\varpi^{-1}(\mathfrak{a}')$. As $\mathfrak{r}/\mathfrak{r}' \cap \mathfrak{a}' = 0$, we see $\boxed{\varpi^{-1}(\mathfrak{a}') \cap \mathfrak{r} = \mathfrak{r}'}$. Also: $\boxed{\mathrm{rad}(\varpi^{-1}(\mathfrak{a}')) = \mathfrak{r}'}$. Indeed, $\mathfrak{r}' \subseteq \varpi^{-1}(\mathfrak{a}')$ is solvable but $\varpi^{-1}(\mathfrak{a}')/\mathfrak{r}' \simeq \mathfrak{a}'$ - s.s.

Applying the induction hypothesis to $\varpi^{-1}(\mathfrak{a}')$, we have $\boxed{\varpi^{-1}(\mathfrak{a}') = \mathfrak{r}' \oplus \underbrace{\mathfrak{a}}_{\text{subalgebra}}}$
(note $\mathfrak{a} \simeq \mathfrak{a}'$ as Lie algebras)

Hence: $\boxed{\mathfrak{g} \simeq \mathfrak{r} \oplus \underbrace{\mathfrak{a}}_{\text{subalgebra}}}$

This completes the induction step, hence also our proof of the theorem. ▥