Talk by Ian Marth (end of class presentation by undergraduate participant)

$\mathbb{K}$ - field of char $=0$, A - finite set, $\mathbb{K}\langle A\rangle$ = free associative algebra on A

$L\langle A\rangle$ - Lie subalgebra of $\langle A\rangle$ generated by A

$\mathbb{K}\langle\langle A\rangle\rangle$ - noncommutative formal power series over A

Thm 1 (BCH f-la): $\forall X, Y \in L\langle A\rangle$, we have:

$$\log(e^X e^Y) = \sum_{k=0}^{\infty} \frac{(-1)^{k-1}}{k} \sum_{m_1+n_1>0} \cdots \sum_{m_k+n_k>0} \frac{1}{\sum(m_i+n_i)} \cdot \frac{[\overbrace{X, \ldots, [X}^{m_1}, \overbrace{[Y, \ldots [Y}^{n_1}, \overbrace{[X, \ldots [X}^{m_2} \ldots, Y \ldots]]}{m_1!\, n_1! \cdots m_k!\, n_k!}$$

Need Hopf algebra structure on $k\langle A\rangle$:

- $\mu: \mathbb{K}\langle A\rangle \otimes \mathbb{K}\langle A\rangle \to k\langle A\rangle$ $\qquad x\otimes y \mapsto xy$
- $\Delta: \mathbb{K}\langle A\rangle \to k\langle A\rangle \otimes k\langle A\rangle$ - alg. hom $\qquad x \mapsto x\otimes 1 + 1\otimes x$ for $x\in A$
- $\varepsilon: \mathbb{K}\langle A\rangle \to \mathbb{K}$ - counit $\qquad \varepsilon(1)=1,\ \varepsilon(x)=0$
- $\alpha: \mathbb{K}\langle A\rangle \to \mathbb{K}\langle A\rangle$ - antipode $\qquad \alpha(1)=1,\ \alpha(a_1 \ldots a_n) = (-1)^n a_n \ldots a_1$

subject to some compatibilities

Theorem 2: Let $D, R: \mathbb{K}\langle A\rangle \to \mathbb{K}\langle A\rangle$ s.t. $D(1)=R(1)=0$, $D(a_1 \ldots a_n) = n\cdot a_1 \ldots a_n$

$R(a_1, \ldots a_n) = [a_1, [a_2, \ldots, [a_{n-1}, a_n] \ldots]]$

For $F\in\mathbb{K}\langle A\rangle$ TFAE:

(1) F is a Lie polynomial

(2) $\Delta(F) = F\otimes 1 + 1\otimes F$

(3) The constant term of F vanishes ($F_0=0$) and $D(F)=R(F)$

(1)$\Leftrightarrow$(2): Friedrichs criteria

(2)$\Leftrightarrow$(3): Dynkin - Specht - Wever criteria

Lemma 1: $\forall F\in k\langle A\rangle$: $\mu\circ(D\otimes\alpha)\circ\Delta(F) = R(F)$

$R(1) = 0 = D(1)$

$$\Delta(a_1 a_2 \ldots a_n) = \sum_{s_1, \ldots s_n \in\{0,1\}} a_1^{s_1} \ldots a_n^{s_n} \otimes a_1^{1-s_1} a_2^{1-s_2} \ldots a_n^{1-s_n}$$

$\Downarrow$

$$\mu\circ(D\otimes\alpha)\circ\Delta(a_1 \ldots a_n) = \sum_{s_1, \ldots s_n \in\{0,1\}} D(a_1^{s_1} \ldots a_n^{s_n})\,\alpha(a_1^{1-s_1} \ldots a_n^{1-s_n}) = \sum_{s_1, \ldots s_n\in\{0,1\}} (-1)^{\sum(1-s_i)} \left(\sum_{i=1}^{n} s_i\right) a_1^{s_1} \ldots a_{n-1}^{s_{n-1}} a_n a_{n-1}^{1-s_{n-1}} \ldots a_1^{1-s_1}$$

$$= \sum_{-\|-} (-1)^{\sum(1-s_i)} s_n a_1^{s_1} \ldots a_{n-1}^{s_{n-1}} a_n a_{n-1}^{1-s_{n-1}} \ldots a_1^{1-s_1} + \underbrace{\sum_{-\|-} (-1)^{\sum(1-s_i)} \left(\sum_{i=1}^{n-1} s_i\right) a_1^{s_1} \ldots a_{n-1}^{s_{n-1}} a_n a_{n-1}^{1-s_{n-1}} \ldots a_1^{1-s_1}}_{=0 \text{ clearly for } s_n\text{-symmetry}}$$

Claim: $\sum_{s_1, \ldots s_{n-1}\in\{0,1\}} (-1)^{\sum_{i=1}^{n-1}(1-s_i)} \cdot \overset{1}{\cancel{s_n}}\, a_1^{s_1} \ldots a_{n-1}^{s_{n-1}} a_n a_{n-1}^{1-s_{n-1}} \ldots a_1^{1-s_1} = \underbrace{R(A)}_{=[a_1,[a_2, \ldots [a_{n-1}, a_n] \ldots]]}$

Obvious (expand brackets in the RHS)

Proof

1) $\Rightarrow$ 2): obvious as being primitive is closed under $[\,,\,]$

2) $\Rightarrow$ 3): $F = (\varepsilon\otimes 1)\Delta(F) = (\varepsilon\otimes 1)(F\otimes 1 + 1\otimes F) = \varepsilon(F) + F \Rightarrow F_0 = 0$

$R(F) = \mu\circ(D\otimes\alpha)\circ\Delta(F) = \mu\circ(D\otimes\alpha)(F\otimes 1 + 1\otimes F) = D(F)$

Lemma 1

(Continuation)

3) $\Rightarrow$ 1) $\quad D(F) = R(F) \Rightarrow F_n = \frac{1}{n} R(F_n) \ \forall n$, $F_0 = 0$ $\Rightarrow$ F – Lie polynomial $\blacksquare$

Lemma 2: $\forall F \in \mathbb{k}\langle A\rangle$: $\underbrace{e^F \text{ is group-like}}_{\Delta(e^F) = e^F \otimes e^F} \iff \underbrace{F\text{ – primitive}}_{\Delta(F) = F\otimes 1 + 1 \otimes F}$

$\Leftarrow$: $\Delta(e^F) = e^{\Delta(F)} = e^{1\otimes F + F\otimes 1} = (e^{1\otimes F})(e^{F\otimes 1}) = (1\otimes e^F)(e^F \otimes 1) = e^F \otimes e^F$

$\Rightarrow$: $\Delta(F) = \Delta(\log(e^F)) = \log(\Delta(e^F)) = \log(e^F \otimes e^F) = \log(\exp(\Delta(F))) = \Delta(($ $\quad 1\otimes F + F\otimes 1$ $\Rightarrow \Delta(F) = F\otimes 1 + 1\otimes F$ $\square$

$X, Y \in L\langle A\rangle \Rightarrow \Delta(e^X e^Y) = e^X e^Y \otimes e^X e^Y \Rightarrow Z := \log(e^X e^Y)$ is primitive

Consider linear map $P: \mathbb{k}\langle A\rangle \to \mathbb{k}\langle A\rangle$ s.t. $P(1) = 0$, $P(a_1 \cdots a_n) = \frac{1}{n} R(a_1 a_2 \ldots a_n)$

If $F \in \mathbb{k}\langle A\rangle$ is primitive, $F = \sum_{n\geq 1} F_n$, then $P(F) = \sum P(F_n)$, $P(F_n) = \frac{1}{n} R(F_n) = \frac{1}{n} D(F_n) = F_n$

Hence: $\log(e^X e^Y) = \sum_{k\geq 1} \frac{(-1)^{k-1}}{k} \left(\sum_{m,n\geq 0} \frac{X^m Y^n}{m!\,n!} - 1\right)^k = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \sum_{\substack{m_1+n_1>0 \\ m_k+n_k>0}} \frac{X^{m_1} Y^{n_1} \ldots X^{m_k} Y^{n_k}}{m_1!\,n_1! \ldots m_k!\,n_k!}$

$$\Big\downarrow P$$

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \sum_{\substack{m_1+n_1>0 \\ \vdots \\ m_k+n_k>0}} \frac{1}{\sum (m_i+n_i)} \cdot \frac{[X, \ldots [X, [Y, \ldots [Y, [X, \ldots \ldots]]}{m_1!\,n_1! \ldots m_k!\,n_k!}$$

This completes the proof of Thm 1! $\blacksquare$