Lecture #30

* First 50min → presentation of the Baker – Campbell – Hausdorff – Dynkin theorem by Ian Martin (see separate 2 page notes).

* Last time: Proof of the Levi theorem

- This result is fundamental both for the proof of Ado's theorem as well as the third fundamental theorem of Lie theory, see Lecture #6.

> **Theorem** (3rd fund thm of Lie theory): Any finite-dimensional Lie algebra over $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$ is the Lie algebra of a Lie group over $\mathbb{K}$.

▸ • Case 1: $\mathfrak{g}$ - semisimple Lie algebra over $\mathbb{K}$

In this case, we know that $\mathrm{Aut}(\mathfrak{g}) \subseteq \mathrm{End}_{\mathrm{line}}(\mathfrak{g})$ is a Lie group with $\mathrm{Lie}(\mathrm{Aut}(\mathfrak{g})) = \mathrm{Der}(\mathfrak{g})$
see [Homework 3, Problem 5(a)]

But for semisimple $\mathfrak{g}$: $\mathrm{Der}(\mathfrak{g})=\mathfrak{g}$, see [Lecture #13, Proposition 2] based on the Cartan's criteria of semisimplicity

Thus: $\mathrm{Aut}(\mathfrak{g})$ is a Lie gp with Lie algebra $\mathfrak{g}$
⇒ its universal cover is a simply connected Lie gp with Lie algebra $\mathfrak{g}$. ✓

• Case 2: $\mathfrak{g}$ - solvable Lie algebra over $\mathbb{K}$

In this case, we have the following result, which implies the desired:

> **Proposition**: If $\mathfrak{g}$ is a finite dimensional Lie algebra over $\mathbb{K}$ of dimension n, then there is a simply connected Lie gp G over $\mathbb{K}$, with $\mathrm{Lie}(G)=\mathfrak{g}$, and G is diffeomorphic to $\mathbb{K}^n$

▸ Proof is by induction on $\dim(\mathfrak{g})$. To run an induction step, we fix any nonzero homomorphism $\chi: \mathfrak{g} \to \mathbb{K}$ (e.g. apply Lie thm to $ad: \mathfrak{g} \curvearrowright \mathfrak{g}$), and set $\bar{\mathfrak{g}} := \mathrm{Ker}(\chi: \mathfrak{g} \to \mathbb{K})$. Then, $\bar{\mathfrak{g}} \subseteq \mathfrak{g}$ - ideal of codim 1, and $\bar{\mathfrak{g}}$ - solvable Lie alg. Furthermore, picking any $d \in \mathfrak{g} \setminus \bar{\mathfrak{g}}$, we can identify

$$\boxed{\mathfrak{g} \simeq \mathbb{K}d \ltimes \bar{\mathfrak{g}}}$$

Let $\bar{G}$ be the simply connected Lie gp with $\mathrm{Lie}(\bar{G})=\bar{\mathfrak{g}}$, which exists by induction assumption (moreover, $\bar{G}$ is diffeomorphic to $\mathbb{K}^{n-1}$)

Lecture #30

(Continuation)

Then: we have a one-parametric family of derivations

$$\boxed{td : \bar{\mathfrak{g}} \to \bar{\mathfrak{g}}\ ,\ t \in \mathbb{K}}$$

which exponentiates to a one-parametric group of automorphisms

$$\boxed{e^{td} : \bar{\mathfrak{g}} \to \bar{\mathfrak{g}}\ ,\ t \in \mathbb{K}}$$

Applying the 2^{nd} Fundamental Theorem of Lie theory, we thus get a one-parameter group of automorphisms

$$\boxed{e^{td} : \bar{G} \to \bar{G}\ ,\ t \in \mathbb{K}}$$

This allows to define the following group structure on $G := \bar{G} \times \mathbb{K}$:

$$\boxed{(x,t)\cdot(y,s) := (x\cdot e^{td}(y),\ t+s) \quad \forall x,y \in \bar{G}\quad \forall s,t \in \mathbb{K}}$$ ← i.e. $G = \mathbb{K} \ltimes \bar{G}$.

Clearly : 1) $Lie(G) = \mathbb{K}d \ltimes Lie(\bar{G}) = \mathbb{K}d \ltimes \bar{\mathfrak{g}} = \mathfrak{g}$

2) G is diffeomorphic to $\mathbb{K}^{n-1} \times \mathbb{K} = \mathbb{K}^n$

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Exercise (important for Ado's thm): If $\mathfrak{g}$ is a nilpotent Lie algebra over $\mathbb{K}$, then $\exp : \mathfrak{g} \to G$ is a diffeomorphism, and the multiplication map written in logarithmic coordinates $\mu : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is polynomial.

- Case 3 : General Case

By Levi decomposition, we have $\boxed{\mathfrak{g} \simeq \mathfrak{g}_{s.s.} \ltimes rad(\mathfrak{g})}$

By Case 1, $\mathfrak{g}_{s.s.}$ admits a simply connected Lie gp which we denote G^{ss}

By Case 2, $rad(\mathfrak{g}) =: \mathfrak{r}$ admits a simply connected Lie gp which we denote T

Once again applying the 2^{nd} Fundamental Theorem of Lie theory, we see that the action of $\boxed{\mathfrak{g}_{s.s} \curvearrowright \underbrace{rad(\mathfrak{g})}_{=\mathfrak{r}}}$ gives rise to the corresponding action of Lie gps : $\boxed{G^{s.s} \curvearrowright T}$

Then: the semidirect product $G := G^{s.s.} \ltimes T$ is the desired Lie gp, as $Lie(G) = \mathfrak{g}_{s.s.} \ltimes \mathfrak{r} = \mathfrak{g}$ and G is simply connected.

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