HOMEWORK 2 (DUE FEBRUARY 16)

Part 1: Pick and write up solutions for any 1 problem among the ones below.

1. Exercise 1 in IV.9 of Kassel's textbook, page 88.
2. Exercise 4 in IV.9 of Kassel's textbook, page 89.

Part 2: Pick and write up solutions for any 1 problem among the ones below.

3. Formula for E^mF^n written in the PBW basis of $U_q(\mathfrak{sl}_2)$ from Lecture 9:

$$E^mF^n = \sum_{i=0}^{\min(m,n)} \begin{bmatrix} m \\ i \end{bmatrix} \begin{bmatrix} n \\ i \end{bmatrix} [i]! \cdot F^{n-i} \cdot \prod_{j=1}^{i} [K; i+j-(m+n)] \cdot E^{m-i} \qquad \forall\ m,n \geq 0.$$

4. Prove the technical exercise used in the proof of Lemma 2 from Lecture 10. Explicitly, if $F^N V = 0$ show that $F^{N-r} h_r V = 0$ for any $0 \leq r \leq N$, where we define

$$h_r := \prod_{j=1-r}^{r-1} [K; r-N+j].$$

Part 3: Pick and write up solutions for any 1 problem among the ones below.

In Problems 5–6, q is assumed to be a primitive d-th root of unity ($d > 2$). We also define

$$e := \begin{cases} d & \text{if } d \text{ is odd} \\ d/2 & \text{if } d \text{ is even} \end{cases}.$$

5. Prove that the center of $U_q(\mathfrak{sl}_2)$ is generated by E^e, F^e, K^e, K^{-e}, and C.

6. Classify all simple finite-dimensional $U_q(\mathfrak{sl}_2)$-modules (cf. Exercise VI.6.3 of Kassel's textbook, page 138).

Part 4 (optional extra problem*): Higher rank versions of $M_q(2), GL_q(2), SL_q(2)$.

Fix $q \in \mathbf{k}^*$ and $n \in \mathbb{Z}_{>0}$. First, we generalize the notion of the quantum plane: the **quantum polynomial algebra** $\mathbf{k}_q[x_1, \ldots, x_n]$ is the associative algebra generated by $x_1, \ldots, x_n$ with the defining relations $x_j x_i = q x_i x_j$ for any $i < j$. Next, we define $M_q(n)$ as the associative algebra generated by $\{t_{i,j}\}_{i,j=1}^n$ with the following defining relations (for any $i < j$ and $a < b$):

$$(\star) \quad t_{j,a}t_{i,a} = qt_{i,a}t_{j,a},\ \ t_{i,b}t_{i,a} = qt_{i,a}t_{i,b},\ \ t_{i,b}t_{j,a} = t_{j,a}t_{i,b},\ \ [t_{i,a}, t_{j,b}] = (q^{-1}-q)t_{i,b}t_{j,a}.$$

The following problem provides an alternative viewpoint towards $M_q(n)$ alike that for $M_q(2)$:

7.1. Given n^2 elements $\{T_{i,j}\}_{i,j=1}^n$ of an algebra R, let us encode them in a single R-valued $n \times n$-matrix $T := \sum_{i,j} T_{i,j} E_{i,j}$. Set $R' := R \otimes \mathbf{k}_q[x_1, \ldots, x_n]$, $R'' := \mathbf{k}_q[x_1, \ldots, x_n] \otimes R$.

Finally, define elements $\{x'_i\}_{i=1}^n$ and $\{x''_i\}_{i=1}^n$ of R' and R'', respectively, via

$$\begin{pmatrix} x'_1 \\ \vdots \\ x'_n \end{pmatrix} = T \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \qquad \text{and} \qquad \big(x''_1, \cdots, x''_n\big) = \big(x_1, \cdots, x_n\big) \cdot T.$$

Assuming $q^2 \neq -1$, prove that the following two conditions are equivalent:

(1) The generators $T_{i,j}$ satisfy the relations $(\star)$ with $t_{i,j}$ replaced by $T_{i,j}$.
(2) We have $x'_j x'_i = q x'_i x'_j$ and $x''_j x''_i = q x''_i x''_j$ for any $i < j$.

Similarly to $M_q(2)$, the above algebra is naturally equipped with a bialgebra structure:

7.2. Verify that $M_q(n)$ is a bialgebra with the coproduct Δ and counit ϵ defined via

$$\Delta(T) = T \otimes T\,, \qquad \epsilon(T) = I_n\,.$$

For what follows, it's instrumental to consider the **quantum skew polynomial algebra** $\Lambda_q[\xi_1, \ldots, \xi_n]$ generated by $\xi_1, \ldots, \xi_n$ with the defining relations $\xi_i^2 = 0,\ \xi_i \xi_j = -q \xi_j \xi_i$ for any $i < j$. The following generalizes Problem 6 from Homework 1:

8. (a) Set $\bar{R} := R \otimes \Lambda_q[\xi_1, \ldots, \xi_n]$ and define $\xi'_1, \ldots, \xi'_n \in \bar{R}$ via $\begin{pmatrix} \xi'_1 \\ \vdots \\ \xi'_n \end{pmatrix} = T \cdot \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix}$. Prove that assertions of Problem 7.1 are equivalent to $x'_j x'_i = q x'_i x'_j, \xi'_i \xi'_j = -q \xi'_j \xi'_i, (\xi'_i)^2 = 0\ \forall\, i < j$.

(b) Find left and right $M_q(n)$-algebra-comodule structures on $\Lambda_q[\xi_1, \ldots, \xi_n]$.

(c) Prove that $\xi'_1 \ldots \xi'_n = \det_q \cdot\, \xi_1 \ldots \xi_n$, where the **quantum determinant** of $M_q(n)$ is

$$\det_q := \sum_{\sigma \in S_n} (-q)^{-l(\sigma)} t_{1,\sigma(1)} \ldots t_{n,\sigma(n)}.$$

(d) Deduce that $\det_q$ is group-like, that is, $\Delta(\det_q) = \det_q \otimes \det_q$.

More generally, given two ordered sets $I = \{i_1, i_2, \ldots, i_k\}$ and $J = \{j_1, j_2, \ldots, j_k\}$ with $1 \leq i_1 < \ldots < i_k \leq n,\ 1 \leq j_1 < \ldots < j_k \leq n$, one defines the **quantum minor** via

$$\tilde{t}^I_J := \sum_{\sigma \in S_k} (-q)^{-l(\sigma)} t_{i_1, j_{\sigma(1)}} \ldots t_{i_k, j_{\sigma(k)}}.$$

9. Recall the two coactions of $M_q(n)$ on $\Lambda_q[\xi_1, \ldots, \xi_n]$ from Problem 8(b).

(a) Write down the formulas for the images of $\xi_I := \xi_{i_1} \ldots \xi_{i_k}$ under both coactions.

(b) Define $\tilde{t}_{j,i} := (-q)^{i-j} \cdot \tilde{t}^{\{1,\ldots,n\}\backslash\{i\}}_{\{1,\ldots,n\}\backslash\{j\}}$. Prove the equalities $\sum_j t_{i,j} \tilde{t}_{j,k} = \delta_{i,k} \cdot \det_q = \sum_j \tilde{t}_{i,j} t_{j,k}$.

(c) Deduce that $\det_q$ is a central element of $M_q(n)$.

In complete analogy with $n = 2$ case, one defines the algebras $GL_q(n)$ and $SL_q(n)$ via:

$$GL_q(n) := M_q(n)[t]/(t\det_q - 1) \qquad \text{and} \qquad SL_q(n) := M_q(n)/(\det_q - 1)\,.$$

Due to Problems 8–9, the bialgebra structure on $M_q(n)$ together with the standard bialgebra structure on $\mathbf{k}[t]$ give rise to bialgebra structures on $GL_q(n)$ and $SL_q(n)$. In fact, we have:

10. Prove that $GL_q(n), SL_q(n)$ are Hopf algebras with the antipode $S(t_{i,j}) = \det_q^{-1} \cdot \tilde{t}_{i,j}$.