HOMEWORK 3 (DUE MARCH 1)

Part 1: Pick and write up solutions for any 1 problem among the ones below.

1. Recall the algebra $\widetilde{U}_q(\mathfrak{sl}_2)$ from Lecture 8, generated by $\{E, F, K^{\pm 1}, L\}$ with a certain list of the defining relations, satisfying the following two properties:

- $\widetilde{U}_q(\mathfrak{sl}_2) \simeq U_q(\mathfrak{sl}_2)$ for $q \neq \pm 1$,
- $\widetilde{U}_{q=1}(\mathfrak{sl}_2)$ is well-defined and is isomorphic to $U(\mathfrak{sl}_2)[K]/(K^2-1)$.

Verify that the resulting algebra isomorphism $\widetilde{U}_{q=1}(\mathfrak{sl}_2)/(K-1) \xrightarrow{\sim} U(\mathfrak{sl}_2)$ is actually a Hopf algebra isomorphism.

2. Let $\mathbf{k}$ be a field of characteristic $\neq 2$, $q \in \mathbf{k}$ be not a root of 1, $\tilde{\Lambda} := \{\pm q^r \mid r \in \mathbb{Z}\}$, and M_1, M_2 be two finite-dimensional $U_q(\mathfrak{sl}_2)$-modules. Recall the $U_q(\mathfrak{sl}_2)$-module intertwiner $\Theta^f \circ \tau \colon M_2 \otimes M_1 \xrightarrow{\sim} M_1 \otimes M_2$ from [Lecture 14, Theorem 1], depending on the choice of

$$f \colon \tilde{\Lambda} \times \tilde{\Lambda} \to \mathbf{k}^\times \quad \text{satisfying} \quad f(\lambda,\mu) = \lambda \cdot f(\lambda, q^2\mu) = \mu \cdot f(q^2\lambda, \mu) \qquad \forall\, \lambda, \mu \in \tilde{\Lambda}.$$

(a) Classify all such maps $f \colon \tilde{\Lambda} \times \tilde{\Lambda} \to \mathbf{k}^\times$.

(b) Classify all such f which in addition satisfy the equalities of [Lecture 15, Proposition 1]:

$$f(\lambda, \mu\nu) = f(\lambda,\mu) f(\lambda,\nu) \quad \text{and} \quad f(\lambda\mu, \nu) = f(\lambda,\nu) f(\mu,\nu) \qquad \forall\, \lambda,\mu,\nu \in \tilde{\Lambda}.$$

Part 2: Pick and write up solutions for any 1 problem among the ones below.

3. Consider algebra automorphisms σ_x and σ_y of the *quantum plane* $\mathbf{k}_q[x,y]$ defined by

$$\sigma_x(x) = qx, \quad \sigma_x(y) = y, \quad \sigma_y(x) = x, \quad \sigma_y(y) = qy$$

as well as linear endomorphisms $\partial_x^{(q)}$ and $\partial_y^{(q)}$ of $\mathbf{k}_q[x,y]$ defined via

$$\partial_x^{(q)}(x^r y^s) = [r] x^{r-1} y^s, \quad \partial_y^{(q)}(x^r y^s) = [s] x^r y^{s-1} \qquad \forall\, r,s \in \mathbb{Z}_{\geq 0}.$$

(a) Verify that the operators

$$E(p) = x \cdot \partial_y^{(q)}(p), \quad F(p) = \partial_x^{(q)}(p) \cdot y, \quad K(p) = \sigma_x \sigma_y^{-1}(p) \qquad \forall\, p \in \mathbf{k}_q[x,y]$$

give rise to an action of the quantum group $U_q(\mathfrak{sl}_2)$ on the quantum plane $\mathbf{k}_q[x,y]$.

(b) Show that the subspace $\mathbf{k}_q[x,y]_n$ of homogeneous degree n elements is a $U_q(\mathfrak{sl}_2)$-submodule. Verify that it is generated by the highest weight vector x^n and is isomorphic to $L(n,+)$.

4. For distinct simple roots $\alpha_i \neq \alpha_j$ of a simple Lie algebra $\mathfrak{g}$, recall $u_{ij}^{\pm} \in \bar{U}_q(\mathfrak{g})$ defined via:

$$u_{ij}^+ := \sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_{q_i} E_i^{1-a_{ij}-r} E_j E_i^r, \qquad u_{ij}^- := \sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_{q_i} F_i^{1-a_{ij}-r} F_j F_i^r.$$

Prove [Lecture 17, Lemma 2] claiming the following two formulas in $\bar{U}_q(\mathfrak{g}) \otimes \bar{U}_q(\mathfrak{g})$:

$$\Delta(u_{ij}^+) = u_{ij}^+ \otimes 1 + K_i^{1-a_{ij}} K_j \otimes u_{ij}^+, \qquad \Delta(u_{ij}^-) = u_{ij}^- \otimes K_i^{-1+a_{ij}} K_j^{-1} + 1 \otimes u_{ij}^-.$$

Part 3: Pick and write up solutions for any 1 problem among the ones below.

5. Generalize the results of Lecture 16 to the higher rank. In other words:

(a) Construct a bialgebra pairing $M(n) \times U(\mathfrak{sl}_n) \to \mathbf{k}$, where $M(n) = \mathbf{k}[Mat_{n\times n}]$.

(b) Verify that it descends to a Hopf pairing $SL(n) \times U(\mathfrak{sl}_n) \to \mathbf{k}$, where $SL(n) = \mathbf{k}[SL_n]$.

6. Generalize the results of Lecture 16 to the q-setup. In other words:

(a) Verify that the pairing $M_q(2) \times U_q(\mathfrak{sl}_2) \to \mathbf{k}$, defined by the same formulas as in the classical case (using matrix coefficients of $U_q(\mathfrak{sl}_2)$-action on $L(1,+)$), is a bialgebra pairing.

(b) Show that it descends to a Hopf pairing $SL_q(2) \times U_q(\mathfrak{sl}_2) \to \mathbf{k}$.

(c) Composing the associated algebra morphism $U_q(\mathfrak{sl}_2) \to SL_q(2)^*$ with the $SL_q(2)^*$-action on $\mathbf{k}_q[x,y]_n^*$ (dual to the $SL_q(2)$-coaction on $\mathbf{k}_q[x,y]_n$ from [Lecture 6, Proposition 2]), verify that one obtains a simple $U_q(\mathfrak{sl}_2)$-module isomorphic to $L(n,+)$.

Part 4 (optional extra problem*)
Ore extensions and application to *PBW-type results* (following Kassel's textbook).

Given an algebra R, let us describe all possible algebra structures on the vector space $R[t]$, which satisfy the following two properties:

(1) the natural inclusion $R \hookrightarrow R[t]$ (given by $x \mapsto x \cdot t^0,\ x \in R$) is an algebra morphism,
(2) $\deg(PQ) = \deg(P) + \deg(Q)$ for any $P, Q \in R[t]$ (we assume $\deg(0) = -\infty$).

7.1. (a) Assume that an algebra structure on $R[t]$ satisfying the above two properties is given. Prove that R has no zero divisors and there exists a unique injective algebra endomorphism α of R and a unique *α-derivation* δ of R (that is, δ is a linear endomorphism of R satisfying $\delta(ab) = \delta(a)b + \alpha(a)\delta(b)$), such that

$$(\star) \qquad ta = \alpha(a)t + \delta(a) \qquad \forall\, a \in R.$$

(b) Prove the inverse statement. Let R be an algebra without zero divisors. Given an injective algebra endomorphism α of R and an α-derivation δ of R, prove that there exists a unique algebra structure on $R[t]$ satisfying the above properties (1), (2), and the formula $(\star)$.

The algebra constructed in 7.1(b) is denoted $R[t;\alpha,\delta]$ and is called the **Ore extension**. A few basic properties of these algebras are summarized in the next two problems.

7.2. In the setup of 7.1(b), show that $R[t;\alpha,\delta]$ has no zero divisors. Verify that as a left R-module, it is free with a basis $\{t^k \,|\, k \in \mathbb{Z}_{\geq 0}\}$. Assuming α to be an automorphism, prove that $R[t;\alpha,\delta]$ is also a right free R-module with the same basis $\{t^k \,|\, k \in \mathbb{Z}_{\geq 0}\}$.

Recall that a ring A is called **left Noetherian** if either of the equivalent conditions hold:

- any left ideal I of A is finitely generated;
- any strictly ascending sequence of left ideals $I_1 \subsetneq I_2 \subsetneq \cdots$ of A is finite.

7.3. Let α be an algebra automorphism of R and δ be an α-derivation of R. Prove that if R is left Noetherian, then so is the Ore extension $R[t;\alpha,\delta]$.

Hint: Given a left ideal I of $R[t;\alpha,\delta]$, consider a collection of subsets $\{I_d\}_{d\in\mathbb{Z}_{\geq 0}}$ of R, consisting of 0 and the leading coefficients of degree d elements of I. Prove that they are left ideals, giving rise to the ascending sequence $I_0 \subseteq \alpha^{-1}(I_1) \subseteq \alpha^{-2}(I_2) \subseteq \cdots$ of left ideals in R.

Recall that a ring A is **Noetherian** if it is left Noetherian and the opposite ring A^{op} is also left Noetherian (equivalently, A is right Noetherian).

7.4. (a) Let R be an algebra without zero divisors, α be the algebra automorphism of R and δ be the α-derivation of R. Verify that $\delta\alpha^{-1}$ is an α^{-1}-derivation of the opposite algebra R^{op} and prove the following algebra isomorphism: $R[t;\alpha,\delta]^{\mathrm{op}} \simeq R^{\mathrm{op}}[t;\alpha^{-1},-\delta\alpha^{-1}]$.

(b) Deduce that $R[t;\alpha,\delta]$ is Noetherian if R is.

The above theory of Ore extensions can be used to deduce the ring-theoretical properties of the algebras we encountered so far in the class: $\mathbf{k}_q[x,y]$, $M_q(2)$, $U_q(\mathfrak{sl}_2)$. The first is straightforward (see 7.5), while the other two require a tower of Ore extensions (see 7.6–7.7).

7.5. Let α be the automorphism of the polynomial ring $\mathbf{k}[x]$ defined via $\alpha(x) = qx$. Verify that $\mathbf{k}_q[x,y] \simeq \mathbf{k}[x][y;\alpha,0]$. Deduce that the quantum plane $\mathbf{k}_q[x,y]$ is Noetherian, has no zero divisors, and the set of monomials $\{x^k y^\ell \,|\, k,\ell \in \mathbb{Z}_{\geq 0}\}$ is its $\mathbf{k}$-basis.

7.6. To present $M_q(2)$ as an iterated Ore extension, we consider the following algebras:

$$A_1 := \mathbf{k}[a], \quad A_2 := \mathbf{k}\langle a,b\rangle/(ba-qab), \quad A_3 := \mathbf{k}\langle a,b,c\rangle/(ba-qab, ca-qac, cb-bc).$$

(a) Let α_1 be the automorphism of A_1 determined by $\alpha_1(a) = qa$. Show that $A_2 \simeq A_1[b;\alpha_1,0]$. Deduce that $\{a^k b^\ell \,|\, k,\ell \in \mathbb{Z}_{\geq 0}\}$ is a $\mathbf{k}$-basis of A_2 (this essentially coincides with 7.5).

(b) Let α_2 be the automorphism of A_2 determined by $\alpha_2(a) = qa,\ \alpha_2(b) = b$. Verify that $A_3 \simeq A_2[c;\alpha_2,0]$. Deduce that $\{a^k b^\ell c^m \,|\, k,\ell,m \in \mathbb{Z}_{\geq 0}\}$ is a $\mathbf{k}$-basis of A_3.

(c) Show that there is a unique algebra automorphism α_3 of A_3 such that $\alpha_3(a) = a,\ \alpha_3(b) = qb,\ \alpha_3(c) = qc$. Verify that the linear endomorphism δ of A_3, defined on the basis by

$$\delta(b^\ell c^m) = 0 \quad \text{and} \quad \delta(a^k b^\ell c^m) = -q^{-1}(1-q^{2k})a^{k-1}b^{\ell+1}c^{m+1} \quad \text{for} \quad k > 0$$

is an α_3-derivation of A_3. Prove that $M_q(2) \simeq A_3[d;\alpha_3,\delta]$.

(d) Deduce that $M_q(2)$ is Noetherian, has no zero divisors, and the set $\{a^k b^\ell c^m d^n\}_{k,\ell,m,n\in\mathbb{Z}_{\geq 0}}$ is a $\mathbf{k}$-basis of $M_q(2)$.

7.7. To present $U_q(\mathfrak{sl}_2)$ as an iterated Ore extension, we consider the following algebras:

$$A_1 := \mathbf{k}[K,K^{-1}], \qquad A_2 := \mathbf{k}\langle K,K^{-1},F\rangle/(K^{\pm 1}\cdot K^{\mp 1}-1, KF-q^{-2}FK).$$

(a) Let α_1 be the automorphism of A_1 determined by $\alpha_1(K) = q^2 K$. Verify that $A_2 \simeq A_1[F;\alpha_1,0]$. Deduce that $\{F^\ell K^m \,|\, \ell \in \mathbb{Z}_{\geq 0}, m \in \mathbb{Z}\}$ is a $\mathbf{k}$-basis of A_2.

(b) Let α_2 be the automorphism of A_2 determined by $\alpha_2(K) = q^{-2}K,\ \alpha_2(F) = F$. Verify that the linear endomorphism δ of A_2, defined on the basis by

$$\delta(K^m) = 0 \quad \text{and} \quad \delta(F^\ell K^m) = F^{\ell-1}\sum_{i=0}^{\ell-1}\frac{q^{-2i}K - q^{2i}K^{-1}}{q-q^{-1}}K^m \quad \text{for} \quad \ell > 0$$

is an α_2-derivation of A_2. Prove that $U_q(\mathfrak{sl}_2) \simeq A_2[E;\alpha_2,\delta]$.

(c) Deduce that $U_q(\mathfrak{sl}_2)$ is Noetherian, has no zero divisors, and the set $\{E^k F^\ell K^m\}^{m\in\mathbb{Z}}_{k,\ell\in\mathbb{Z}_{\geq 0}}$ is a $\mathbf{k}$-basis of $U_q(\mathfrak{sl}_2)$.