HOMEWORK 4 (DUE MARCH 15)

Part 1: Pick and write up solutions for any 2 problems among the ones below.

1. Verify the following equality for $n \in \mathbb{Z}_{>0}$ (used in the proof of [Lecture 19, Lemma 4]):

$$\sum_{r=0}^{n} (-1)^n \begin{bmatrix} n \\ r \end{bmatrix}_q q^{\pm(1-n)r} = 0 \,.$$

2. Verify that the following gives an action of a Hopf algebra H on itself (**adjoint action**):

$$\mathrm{ad}\colon H \curvearrowright H \qquad \text{given by} \qquad \mathrm{ad}(x)y = \sum_{(x)} x' y S(x'') \,.$$

3. For any collections $p_i \in \mathbb{Z}$, $r_i, s_i \in \mathbb{Z}_{>0}$ (with $i \in I$–the indexing set of simple roots of $\mathfrak{g}$) show that the operators E_i, F_i act locally nilpotently on the $U_q(\mathfrak{g})$-module $U_q(\mathfrak{g})/J$, where J is the left ideal of $U_q(\mathfrak{g})$ generated by $E_i^{r_i}, F_i^{s_i}, K_i - q_i^{p_i}$ (this is [Lectures 20-21, Lemma 6]).

4. Show that if $u \in U_q(\mathfrak{g})$ acts trivially on all finite-dimensional $U_q(\mathfrak{g})$-modules, then $u = 0$.
Hint: Try to argue similarly to our proof of [Lecture 18, Theorem 1], i.e. consider actions of $U_q(\mathfrak{g})$ on the tensor products of $\widetilde{L}(\lambda)$ and their twist by Cartan involution for $\lambda \gg 0$.

Part 2: Pick and write up solutions for any 2 problems among the ones below.

5. Recall the setup of Lecture 22: $K = \mathbb{Q}(v) \supset \mathbb{Q}[v, v^{-1}] = A$, V is either $L(\lambda)$ or $\widetilde{L}(\lambda)$ with λ being a dominant integral weight, and $V_A := \sum_{\mathfrak{J}} A F_{\mathfrak{J}} v_\lambda$. Verify that V_A is a free A-module, such that $V_A \otimes_A K \to V$ is an isomorphism, and show $\dim_K(V) = \mathrm{rk}_A(V_A) = \dim_{\mathbb{C}}(V \otimes_A \mathbb{C})$.

6. Complete the arguments in our proof of [Lecture 22, Theorem 2] by proving:

(a) $L(\lambda)^* \simeq L(-w_0\lambda)$ for any dominant integral weight λ.
Hint: If you don't know Weyl groups, treat $\mathfrak{g} = \mathfrak{sl}_n$ case with $w_0(\lambda_1, \ldots, \lambda_n) = (\lambda_n, \ldots, \lambda_1)$.

(b) $V \simeq (V^*)^*$ as $U_q(\mathfrak{g})$-modules for any finite-dimensional module V.
Hint: The usual vector space isomorphism $V \xrightarrow{\sim} (V^*)^*$ from linear algebra does not intertwine $U_q(\mathfrak{g})$-actions as the antipode is not involutive, but it can be fixed with a help of U_q^0.

7. Consider an embedding $\mathbb{Q}(v) \hookrightarrow \mathbf{k}$ given by $v \mapsto q \in \mathbf{k}$, where q is transcendental over $\mathbb{Q}$. For a dominant integral weight λ, verify that $L(\lambda)_{\mathbb{Q}(v)} \otimes_{\mathbb{Q}(v)} \mathbf{k}$ is a simple module over $U_{\mathbb{Q}(v)} \otimes_{\mathbb{Q}(v)} \mathbf{k} \simeq U_{\mathbf{k}}$. Deduce the equality $L(\lambda)_{\mathbb{Q}(v)} \otimes_{\mathbb{Q}(v)} \mathbf{k} \simeq L(\lambda)_{\mathbf{k}}$.

8. (a) Generalize the results on $\widetilde{U}_q(\mathfrak{sl}_2)$ of Lecture 8 by defining an algebra $\widetilde{U}_q(\mathfrak{g})$ generated by $\{E_i, F_i, K_i^{\pm 1}, L_i\}_{i \in I}$ with an explicit list of the defining relations such that
∘ If $q^{2d_i} \neq 1 \ \forall\, i \in I$, then the assignment $E_i \mapsto E_i, F_i \mapsto F_i, K_i \mapsto K_i$ gives rise to an isomorphism $U_q(\mathfrak{g}) \xrightarrow{\sim} \widetilde{U}_q(\mathfrak{g})$,
∘ For $q = 1$, we have an isomorphism $\widetilde{U}_{q=1}(\mathfrak{g}) \simeq U(\mathfrak{g})[\{K_i\}_{i \in I}]/(\{K_i^2 - 1\}_{i \in I})$.

(b) Show that the latter induces a Hopf algebra isomorphism $\widetilde{U}_{q=1}(\mathfrak{g})/(\{K_i - 1\}_{i \in I}) \simeq U(\mathfrak{g})$.

Part 3 (optional extra problems*)

9. *Explicit construction of representations $L(\lambda)$ for minuscule dominant weights λ.*

Recall that a nonzero dominant integral weight λ is called *minuscule* if $(\lambda,\alpha)\in\{-d_\alpha,0,d_\alpha\}$ for any root α, where $d_\alpha:=\frac{(\alpha,\alpha)}{2}$.

(a) Verify that the weights of the simple $U_q(\mathfrak{g})$-module $L(\lambda)$ with the highest weight λ are precisely the conjugates of λ under the Weyl group W, each occurring with a multiplicity 1.

Let us now provide an explicit construction of such $L(\lambda)$. Consider a vector space L with a basis $\{x_\mu\}_{\mu\in W(\lambda)}$. We define endomorphisms e_i,f_i,k_i $(i\in I)$ of L as follows:

$$k_i(x_\mu)=q^{(\alpha_i,\mu)}x_\mu,\ e_i(x_\mu)=\begin{cases}x_{s_i(\mu)} & \text{if } (\mu,\alpha_i)=-d_{\alpha_i}\\ 0 & \text{otherwise}\end{cases},\ f_i(x_\mu)=\begin{cases}x_{s_i(\mu)} & \text{if } (\mu,\alpha_i)=d_{\alpha_i}\\ 0 & \text{otherwise}\end{cases}$$

for any $\mu\in W(\lambda)$.

(b) Show that the assignment $E_i\mapsto e_i, F_i\mapsto f_i, K_i\mapsto k_i$ defines an action of $U_q(\mathfrak{g})$ on L.

(c) Prove that $L\simeq L(\lambda)$. Verify that L is simple even if q is a root of unity.

(d) Derive explicit formulas for the *vector representations* in the classical types A_n,B_n,C_n,D_n.

10. *Explicit construction of quantum analogues of the adjoint representations.*

Consider a vector space L with a basis $\{x_\gamma\}_{\gamma\in\Delta}\cup\{h_i\}_{i\in I}$, where Δ denotes the root system of $\mathfrak{g}$. Define endomorphisms k_i of L via $k_i(h_j)=h_j, k_i(x_\gamma)=q^{(\alpha_i,\gamma)}x_\gamma$. Next, we define endomorphisms e_i,f_i of L as follows:

- $e_i(x_{\alpha_i})=0,\ e_i(x_{-\alpha_i})=h_i,\ e_i(h_i)=[2]_{q_i}x_{\alpha_i},\ e_i(h_j)=\Big[-d_j^{-1}(\alpha_i,\alpha_j)\Big]_{q_j}\cdot x_{\alpha_i}$ for $j\neq i$,
- $f_i(x_{\alpha_i})=h_i,\ f_i(x_{-\alpha_i})=0,\ f_i(h_i)=[2]_{q_i}x_{-\alpha_i},\ f_i(h_j)=\Big[-d_j^{-1}(\alpha_i,\alpha_j)\Big]_{q_j}\cdot x_{-\alpha_i}$ for $j\neq i$,
- All the remaining roots $\Delta\backslash\{\pm\alpha_i\}$ split into α_i-*strings* of the form $\{\gamma,\gamma-\alpha_i,\ldots,\gamma-m\alpha_i\}$ with $\gamma+\alpha_i,\gamma-(m+1)\alpha_i\notin\Delta$ (note that $m=d_{\alpha_i}^{-1}\cdot(\gamma,\alpha_i)$). For each such α_i-string, define

$$e_i(x_{\gamma-k\alpha_i})=\begin{cases}[m+1-k]_{q_i}\cdot x_{\gamma-(k-1)\alpha_i} & \text{if } 0<k\le m\\ 0 & \text{if } k=0\end{cases},$$

$$f_i(x_{\gamma-k\alpha_i})=\begin{cases}[k+1]_{q_i}\cdot x_{\gamma-(k+1)\alpha_i} & \text{if } 0\le k<m\\ 0 & \text{if } k=m\end{cases}.$$

Verify that the assignment $E_i\mapsto e_i$, $F_i\mapsto f_i$, $K_i\mapsto k_i$ defines an action of $U_q(\mathfrak{g})$ on L (this is the quantum analogue of the *adjoint representation*).