HOMEWORK 5 (DUE APRIL 5)

Part 1: Pick and write up solutions for any 1 problem among the ones below.

1. Prove [Lecture 25, Lemma 5], that is, verify

$$\mathrm{ad}(E_i)(yK_\lambda x) = yK_\lambda(q^{-(\lambda,\alpha_i)}E_i x - q^{(\mu-\nu,\alpha_i)}xE_i) + \frac{(q^{-(\nu-\alpha_i,\alpha_i)}r_i(y)K_{\lambda+\alpha_i} - r'_i(y)K_{\lambda-\alpha_i})x}{q_i - q_i^{-1}},$$

$$\mathrm{ad}(F_i)(yK_\lambda x) = q^{-(\mu,\alpha_i)}(F_i y - q^{-(\lambda,\alpha_i)}yF_i)K_{\lambda+\alpha_i}x + \frac{y(q^{-(\mu-\alpha_i,\alpha_i)}K_\lambda r'_i(x) - q^{-2(\mu-\alpha_i,\alpha_i)}K_{\lambda+2\alpha_i}r_i(x))}{q_i - q_i^{-1}}$$

for any $i \in I$, $\lambda \in Q$, $x \in (U_q^+)_\mu$, $y \in (U_q^-)_{-\nu}$.

2. Complete the proof of [Lecture 26, Proposition 1] by verifying

$$\langle \mathrm{ad}(F_i)v, v' \rangle = \langle v, \mathrm{ad}(S(F_i))v' \rangle$$

for any $i \in I$ and $v, v' \in U_q(\mathfrak{g})$.

Part 2: Pick and write up solutions for any 1 problem among the ones below.

3. Verify that Proposition 1 of Lecture 26 is equivalent to the following result:

$$\langle \cdot, \cdot \rangle \colon U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g}) \to \mathbf{k} \qquad \text{is a } U_q(\mathfrak{g}) - \text{module morphism.}$$

4. Complete our argument in [Lecture 27, Proposition 1] by verifying that

$$u \mapsto \mathrm{tr}_{L(\lambda)}(uK_{-2\rho}) \quad \text{gives rise to a } U_q(\mathfrak{g}) - \text{module morphism} \quad U_q(\mathfrak{g}) \to \mathbf{k}.$$

Part 3: Pick and write up solutions for any 2 problems among the ones below.

5. Complete the proof of [Lecture 28, Lemma 2] by verifying the following equality:

$$(1 \otimes F_j)\Theta_\mu + (F_j \otimes K_j^{-1})\Theta_{\mu-\alpha_j} = \Theta_\mu(1 \otimes F_j) + \Theta_{\mu-\alpha_j}(F_j \otimes K_j).$$

6. Complete the proof of [Lecture 28, Lemma 3] by verifying the following equality:

$$(\mathrm{id} \otimes \Delta)\Theta_\mu = \sum_{0 \le \nu \le \mu} (\Theta_{\mu-\nu})_{12}(1 \otimes K_\nu \otimes 1)(\Theta_\nu)_{13}.$$

7. Assuming $\mathbf{k}$ contains appropriate roots of q, determine all maps $f \colon P \times P \to \mathbf{k}$ satisfying

$$f(\lambda+\eta, \mu) = q^{-(\eta,\mu)}f(\lambda, \mu), \ \ f(\lambda, \mu+\eta) = q^{-(\eta,\lambda)}f(\lambda, \mu),$$
$$f(\lambda+\nu, \mu) = f(\lambda, \mu)f(\nu, \mu), \ \ f(\lambda, \mu+\nu) = f(\lambda, \mu)f(\lambda, \nu)$$

for any $\lambda, \mu, \nu \in P$ and $\eta \in Q$.

8. Consider the simplest case $\mathfrak{g} = \mathfrak{sl}_2, M = M' = V = L(1,+)$ with $f \colon \mathbb{Z} \times \mathbb{Z} \to \mathbf{k}$ determined by $f(1,1) = q^{-1}$, and let $R = \Theta^f \circ \tau$.

(a) Verify that $R^2 = (q^{-1} - q)R + 1$, or equivalently, $(qR^{-1})^2 = (q^2-1)(qR^{-1}) + q^2$.

(b) Deduce that the operators $\{R'_i\}_{i=1}^{r-1} \subset \mathrm{End}(V^{\otimes r})$ given by $R'_i = qR_i^{-1}$ satisfy

$$R'_i R'_{i+1} R'_i = R'_{i+1} R'_i R'_{i+1}, \ \ R'_i R'_j = R'_j R'_i \ (j \neq i, i \pm 1), \ \ (R'_i)^2 = (q^2-1)R'_i + q^2.$$

In other words, $\{R'_i\}_{i=1}^{r-1}$ define a representation of the type A_{r-1} Hecke algebra on $V^{\otimes r}$.