HOMEWORK 6 (DUE APRIL 19)

Part 1: Pick and write up solutions for <u>any 1</u> problem among the ones below.

1. Endow $k_q[G]$ from Lecture 29 with a natural Hopf algebra structure.

2. In the simplest case $G = \mathrm{SL}(2)$, verify:

(a) $k_q[\mathrm{SL}(2)]$ is generated by the matrix coefficients corresponding to $M = L(1,+)$.

(b) the corresponding four matrix coefficients from (a) satisfy the defining relations of $\mathrm{SL}_{q^{-1}}(2)$ from Lecture 6, thus giving rise to a surjective homomorphism $\phi\colon \mathrm{SL}_{q^{-1}}(2) \to k_q[\mathrm{SL}(2)]$.

(c) the homomorphism ϕ from (b) is actually a Hopf algebra isomorphism.

Part 2: Pick and write up solutions for <u>any 2</u> problems among the ones below.

3. (a) Verify the following algebraic equality:

$$\sum_{0\le a,c\le i} (-1)^{b+j} q^{\pm(b-ac-j(i+1))} \begin{bmatrix} i \\ a \end{bmatrix}_q \begin{bmatrix} j+c \\ c \end{bmatrix}_q \begin{bmatrix} j+a \\ i-c \end{bmatrix}_q = 1 \text{ with } b = a+c+j-i.$$

Hint: You may want to prove first $\begin{bmatrix} a+b \\ k \end{bmatrix}_q = \sum_{i=0}^k q^{ai-b(k-i)} \begin{bmatrix} b \\ i \end{bmatrix}_q \begin{bmatrix} a \\ k-i \end{bmatrix}_q$ for any $a,b,k \ge 0$.

(b) Complete the proof of [Lecture 30, Lemma 1(a, b)].

4. Prove [Lecture 31, Lemma 2] by induction on p (the base case $p = 1$ treated in class).

5. Given $i \ne j \in I$, a finite-dimensional $U_q(\mathfrak{g})$-module V of type 1, and $v \in V$, verify that

$$T_i(E_j v) = \big(\mathrm{ad}(E_i^{(r)})E_j\big)(T_i(v)) \qquad \text{where } r := -2(\alpha_i,\alpha_j)/(\alpha_i,\alpha_i),$$

thus establishing the first equality of [Lecture 31, Proposition 1].

6. Let σ denote the antiautomorphism of $U_q(\mathfrak{g})$ from Lecture 28, and consider an algebra homomorphism ${}^{\sigma}\mathrm{ad}\colon U_q(\mathfrak{g}) \to \mathrm{End}_{\mathbf{k}}(U_q(\mathfrak{g}))$ given by $x \mapsto \sigma \circ \mathrm{ad}(x) \circ \sigma$.

(a) Prove $T_i({}^{\sigma}\mathrm{ad}(E_i)u) = \mathrm{ad}(F_i)T_i(u)$ and $T_i({}^{\sigma}\mathrm{ad}(F_i)u) = \mathrm{ad}(E_i)T_i(u)$ for any $u \in U_q(\mathfrak{g})$.

(b) Prove $T_i({}^{\sigma}\mathrm{ad}(E_i^{(m)})E_j) = \mathrm{ad}(E_i)^{(r-m)}E_j$ for any $i \ne j$ and $r = -2(\alpha_i,\alpha_j)/(\alpha_i,\alpha_i)$.

(c) Derive explicit formulas for $T_i^{-1}(E_j)$ and $T_i^{-1}(F_j)$ stated in the end of Lecture 31.

Part 3: Pick and write up solutions for <u>any 1</u> problem among the ones below.

7. Given $i \ne j \in I$ such that $s_i s_j \in W$ is of order 4, prove $T_iT_jT_iT_j = T_jT_iT_jT_i$ in $\mathrm{Aut}(U_q(\mathfrak{g}))$, thus establishing the corresponding case of [Lecture 32, Theorem 1].

8. Given $i \ne j \in I$ such that $s_i s_j \in W$ is of order 4, and $w \in \langle s_i, s_j\rangle \subset W$ satisfying $w(\alpha_i) > 0$, verify that $T_w(E_i) \in \langle E_i, E_j\rangle \subset U_q^+$, and furthermore $T_w(E_i) = E_k$ if $w(\alpha_i) = \alpha_k$, thus establishing the corresponding case of [Lecture 32, Lemma 1].

9. Given $i \ne j \in I$ such that $s_i s_j \in W$ is of order 4, verify that

$$\mathrm{span}\,\{T_iT_jT_i(E_j)^{a_4} \cdot T_iT_j(E_i)^{a_3} \cdot T_i(E_j)^{a_2} \cdot E_i^{a_1} \,|\, a_1,a_2,a_3,a_4 \ge 0\} = \langle E_i, E_j\rangle,$$

thus establishing the corresponding case of [Lecture 33, Lemma 1].