

Lecture #1

\* Algebras

Recall that an algebra  $A$  is a ring together with a ring map ("unit")  $\eta: \mathbb{k} \rightarrow A$  whose image is in the center of  $A$  (here,  $\mathbb{k}$  is the ground field). Thus,  $A$  is a vector space  $V_{\mathbb{k}}$  with two maps  $\mu: A \otimes A \rightarrow A$  and  $\eta: \mathbb{k} \rightarrow A$  satisfying

$$\begin{array}{ccc}
 A \otimes A \otimes A & \xrightarrow{\mu \otimes \text{id}} & A \otimes A \\
 \downarrow \text{id} \otimes \mu & & \downarrow \mu \\
 A \otimes A & \xrightarrow{\mu} & A
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccccc}
 \mathbb{k} \otimes A & \xrightarrow{\eta \otimes \text{id}} & A \otimes A & \xleftarrow{\text{id} \otimes \eta} & A \otimes \mathbb{k} \\
 & \searrow \sim & \downarrow \mu & \swarrow \sim & \\
 & & A & & 
 \end{array}$$

Recall that an algebra  $A$  is called commutative if  $ab=ba \forall a,b \in A$ , i.e.

$$\begin{array}{ccc}
 A \otimes A & \xrightarrow{\text{flip } \tau_{A,A}} & A \otimes A \\
 \mu \searrow & & \swarrow \mu \\
 & A & 
 \end{array}$$

A morphism of algebras is a linear map  $f: A \rightarrow B$  intertwining  $\mu$  and  $\eta$

$$\begin{array}{ccc}
 \mathbb{k} & \xrightarrow{\eta_A} & A & \xrightarrow{f} & B \\
 & & \mu_B & & \\
 & & \text{and} & & 
 \end{array}
 \quad
 \begin{array}{ccc}
 A \otimes A & \xrightarrow{f \otimes f} & B \otimes B \\
 \downarrow \mu_A & & \downarrow \mu_B \\
 A & \xrightarrow{f} & B
 \end{array}$$

If  $A$  is an algebra, then the opposite algebra  $A^{op}$  is the same v. space but with  $\mu_{A^{op}}(a,b) = \mu_A(b,a)$ , i.e.  $\mu_{A^{op}} = \mu_A \circ \tau_{A,A}$  (flip map)

The center of an algebra  $A$  is  $Z(A) = \{a \in A \mid ab=ba \forall b \in A\}$

The 2-sided ideal is a subspace  $I \subseteq A$  s.t.  $\mu(I \times A), \mu(A \times I) \subseteq I$

Lemma 1: If  $I$  is a 2-sided ideal of  $A$ , then there is a unique algebra structure on the quotient vector space  $A/I$  such that the canonical projection  $A \rightarrow A/I$  is an algebra morphism.

Easy exercise: prove this lemma.

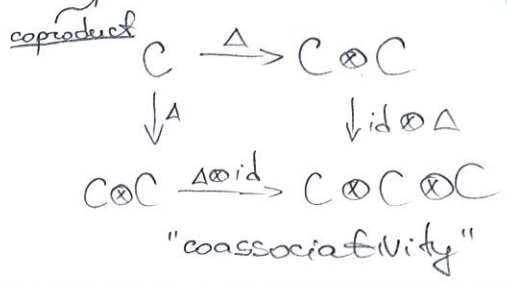
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\* Coalgebras

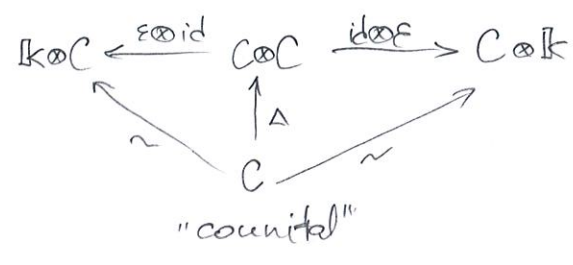
The notion of coalgebras is "dual" to algebras, and will be obtained by reversing all arrows

Def: A coalgebra is a triple  $(C, \Delta, \epsilon)$ , where  $C$  is a vector space and

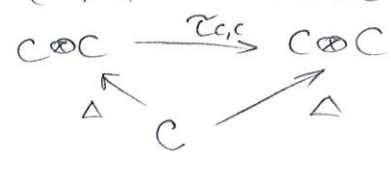
$\Delta: C \rightarrow C \otimes C$ ,  $\epsilon: C \rightarrow k$  are linear maps satisfying:



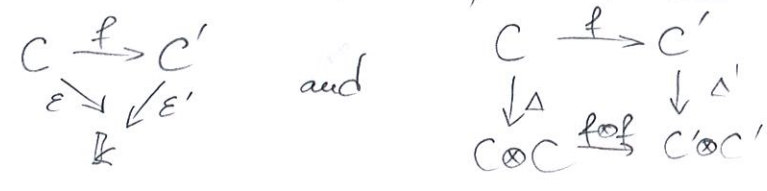
and



Def: A coalgebra  $(C, \Delta, \epsilon)$  is called cocommutative if



Def: Given two coalgebras  $(C, \Delta, \epsilon)$  and  $(C', \Delta', \epsilon')$ ; a linear map  $f: C \rightarrow C'$  is a morphism of coalgebras if the following commute:



• If  $C$  is a coalgebra (with coproduct  $\Delta$ , counit  $\epsilon$ ), then the opposite coalgebra  $C^{op}$  is the same vector space with  $\Delta^{op} = \tau_{C,C} \circ \Delta$  and  $\epsilon^{op} = \epsilon$

Def: A subspace  $I$  of a coalgebra  $(C, \Delta, \epsilon)$  is called a coideal if  $\Delta(I) \subseteq I \otimes C + C \otimes I$  and  $\epsilon(I) = 0$

Lemma 2: If  $I$  is a coideal of a coalgebra  $C$ , then there is a unique (easy exercise) coalgebra structure on the quotient space  $C/I$  such that the canonical projection  $C \rightarrow C/I$  is a coalgebra morphism.

This  $C/I$  is called the quotient coalgebra.

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There is a simple relation between algebras and coalgebras:

Lemma 3: a) The dual vector space of a coalgebra is an algebra

b) The dual vector space of a finite-dimensional algebra is a coalgebra

▶ a) Dualizing  $\Delta$ , we get  $\Delta^*: (C \otimes C)^* \rightarrow C^*$ .

On the other hand, we always have a natural map  $\lambda: C^* \otimes C^* \rightarrow (C \otimes C)^*$ .

Composing, we see that  $(C^*, \Delta^* \circ \lambda, \varepsilon^*)$  is an algebra (check details!)

b) If  $A$  is a fin. dim. algebra, then dualizing  $\mu$  we get  $\mu^*: A^* \rightarrow (A \otimes A)^*$  and  $(A \otimes A)^* \xrightarrow{\lambda^{-1}} A^* \otimes A^*$  given that  $\dim(A) < \infty$ . Hence,  $(A^*, \lambda^{-1} \circ \mu^*, \eta^*)$  is a coalgebra (check details!)

Sweedler's Notation: For an element  $x$  of a coalgebra  $(C, \Delta, \varepsilon)$  we shall write

$$\Delta(x) = \sum_{(x)} x' \otimes x'' \in C \otimes C$$

Then the coassociativity reads as follows:

$$\sum_{(x)} \left( \sum_{(x')} (x')' \otimes (x')'' \right) \otimes x'' = \sum_{(x)} x' \otimes \left( \sum_{(x'')} (x'')' \otimes (x'')'' \right)$$

and will be rather written as

$$\sum_{(x)} x' \otimes x'' \otimes x''' \quad \text{or} \quad \sum_{(x)} x^{(1)} \otimes x^{(2)} \otimes x^{(3)}$$

Generalizing this, we define  $\Delta^{(n)}: C \rightarrow C^{\otimes(n+1)}$  inductively via

$$\Delta^{(1)} = \Delta, \quad \Delta^{(n)} = (\Delta \otimes \text{id}_{C^{\otimes n}}) \circ \Delta^{(n-1)} \quad \text{for } n > 1$$

Then we shall write  $\Delta^{(n)}(x) = \sum_{(x)} x^{(1)} \otimes x^{(2)} \otimes \dots \otimes x^{(n+1)}$

On the other hand, the counitality condition reads

$$\sum_{(x)} \varepsilon(x') \cdot x'' = x = \sum_{(x)} x' \cdot \varepsilon(x'')$$

while the cocounitality becomes

$$\sum_{(x)} x' \otimes x'' = \sum_{(x)} x'' \otimes x'$$

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Examples

1) Let  $X$  be any set, set  $C = k[X] = \bigoplus_{x \in X} kx$  with  $\Delta(x) = x \otimes x$ ,  $\epsilon(x) = 1 \quad \forall x \in X$   
 This is coalgebra of a set.

[Exercise: Show that the dual algebra of  $k[X]$  is the algebra of  $k$ -valued  $f$ -s on  $X$ .

2) Given two coalgebras  $(C, \Delta, \epsilon)$  and  $(C', \Delta', \epsilon')$ , the tensor product  $C \otimes C'$  is naturally equipped with coalgebra structure with  
 $\epsilon_{C \otimes C'} = \epsilon \otimes \epsilon'$  and  $\Delta_{C \otimes C'} = (\text{id} \otimes \tau_{C, C'} \otimes \text{id})(\Delta \otimes \Delta')$

This is the tensor product of coalgebras.

[Exercise: For any two sets  $X, Y$  show that  $k[X] \otimes k[Y] \cong k[X \times Y]$ .

3) The dual of  $A = M_n(k) = \{ \text{nxn matrices over } k \}$  is called the matrix coalgebra

[Exercise: Let  $\{x_{ij} \mid i, j = 1, \dots, n\}$  be the basis dual to  $\{E_{ij}\}$ . Verify that  
 $\epsilon(x_{ij}) = \delta_{ij}$  and  $\Delta(x_{ij}) = \sum_k x_{ik} \otimes x_{kj}$ .

\* Bialgebras

Let us now assume that a vector space  $H$  is equipped both with an algebra structure  $(H, \mu, \eta)$  and a coalgebra structure  $(H, \Delta, \epsilon)$ . Then, the tensor product  $H \otimes H$  is equipped both with algebra and coalgebra structure.

Proposition 1: TFAE  
 (i) The maps  $\mu, \eta$  are morphisms of coalgebras  
 (ii) The maps  $\Delta, \epsilon$  are morphisms of algebras.

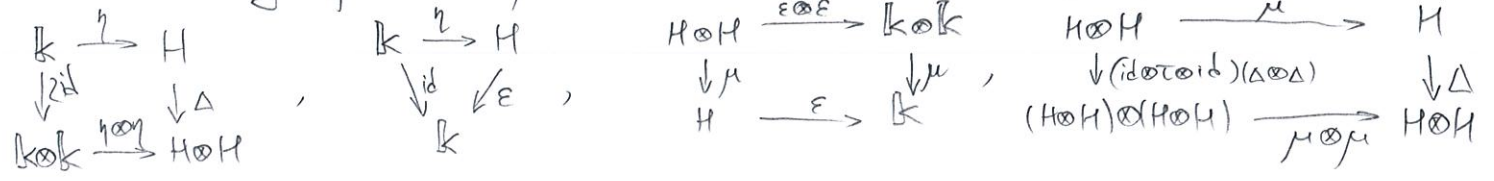
Def: A bialgebra is a quadruple  $(H, \mu, \eta, \Delta, \epsilon)$  where  $(H, \mu, \eta)$ -algebra,  $(H, \Delta, \epsilon)$ -coalgebra, which satisfy the equivalent conditions of Prop 1

Def: A morphism of bialgebras is a linear map, which is both a morphism of underlying algebras and coalgebras

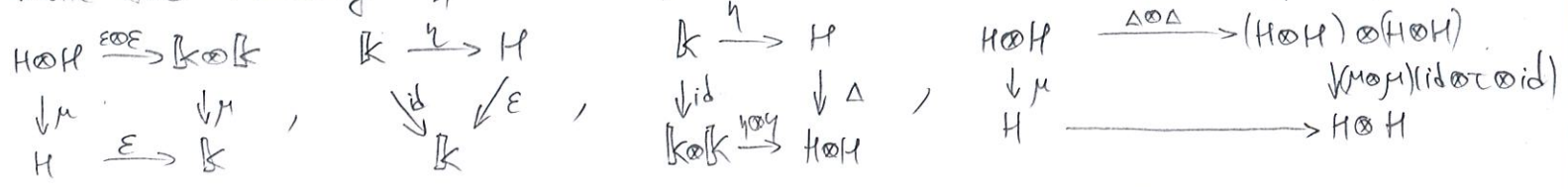
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Proof of Proposition 1

The validity of (i) is equivalent to the commutativity of 4 diagrams:



while the validity of (ii) is equivalent to the commutativity of:



Clearly, these 4 diagrams are the same as the first four.

Thus, for bialgebras we have the following simple compatibilities

$$\varepsilon(1) = 1, \quad \Delta(1) = 1 \otimes 1, \quad \varepsilon(xy) = \varepsilon(x)\varepsilon(y), \quad \Delta(xy) = \Delta(x)\Delta(y)$$

Exercise: If  $H = (H, \mu, \eta, \Delta, \varepsilon)$  is a bialgebra, then the following are also bialgs:

- $H^{\text{op}} = (H, \mu^{\text{op}}, \eta, \Delta, \varepsilon)$
- $H^{\text{cop}} = (H, \mu, \eta, \Delta^{\text{op}}, \varepsilon)$
- $H^{\text{op, cop}} = (H, \mu^{\text{op}}, \eta, \Delta^{\text{op}}, \varepsilon)$

As an immediate corollary of Lemma 3, we also get:

Corollary 1: A dual of a finite-dimensional bialgebra has a canonical bialgebra structure

Q: Which extra structure on a set  $X$  makes  $k[X]$  into a bialgebra?  
 (e.g. an associative map  $\mu: X \times X \rightarrow X$  with a left & right unit  $e$ ).