# Lecture #1

## * Algebras

- Recall that an *algebra* $A$ is a ring together with a ring map ("unit") $\eta: \Bbbk \to A$ whose image is in the center of $A$ (here, $\Bbbk$ is the ground field). Thus, $A$ is a vector space $/\Bbbk$ with two maps $\mu: A \otimes A \to A$ and $\eta: \Bbbk \to A$ satisfying

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "\mu \otimes id"] \arrow[d, "id \otimes \mu"'] & A \otimes A \arrow[d, "\mu"] \\ A \otimes A \arrow[r, "\mu"] & A \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \Bbbk \otimes A \arrow[r, "\eta \otimes id"] \arrow[dr, "\sim"'] & A \otimes A \arrow[d, "\mu"] & A \otimes \Bbbk \arrow[l, "id \otimes \eta"'] \arrow[dl, "\sim"] \\ & A & \end{tikzcd}$$

- Recall that an algebra $A$ is called *commutative* if $ab = ba$ $\forall a, b \in A$, i.e.

$$\begin{tikzcd} A \otimes A \arrow[rr, "flip\ \tau_{A,A}"] \arrow[dr, "\mu"'] & & A \otimes A \arrow[dl, "\mu"] \\ & A & \end{tikzcd}$$

- A *morphism of algebras* is a linear map $f: A \to B$ intertwining $\mu$ and $\eta$

$$\begin{tikzcd} \Bbbk \arrow[r, "\eta_A"] \arrow[rr, bend right, "\mu_B"'] & A \arrow[r, "f"] & B \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A \otimes A \arrow[r, "f \otimes f"] \arrow[d, "\mu_A"'] & B \otimes B \arrow[d, "\mu_B"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

- If $A$ is an algebra, then the *opposite algebra* $A^{op}$ is the same v. space but with $\mu_{A^{op}}(a, b) = \mu_A(b, a)$, i.e. $\mu_{A^{op}} = \mu_A \circ \underbrace{\tau_{A,A}}_{\text{flip map}}$

- The *center* of an algebra $A$ is $Z(A) = \{a \in A \mid ab = ba \ \forall\, b \in A\}$

- The *2-sided ideal* is a subspace $I \subseteq A$ s.t. $\mu(I \times A), \mu(A \times I) \subseteq I$

> **Lemma 1**: If $I$ is a 2-sided ideal of $A$, then there is a unique algebra structure on the quotient vector space $A/I$ such that the canonical projection $A \to A/I$ is an algebra morphism.

Easy exercise: prove this lemma.

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## * Coalgebras

The notion of coalgebras is "dual" to algebras, and will be obtained by reversing all arrows

**Def**: A *coalgebra* is a triple $(C, \Delta, \varepsilon)$, where $C$ is a vector space and

$\Delta: C \to C \otimes C$ (coproduct), $\varepsilon: C \to \Bbbk$ (counit) are linear maps satisfying:

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "id \otimes \Delta"] \\ C \otimes C \arrow[r, "\Delta \otimes id"] & C \otimes C \otimes C \end{tikzcd}$$

"coassociativity"

and

$$\begin{tikzcd} \Bbbk \otimes C & C \otimes C \arrow[l, "\varepsilon \otimes id"'] \arrow[r, "id \otimes \varepsilon"] & C \otimes \Bbbk \\ & C \arrow[ul, "\sim"] \arrow[u, "\Delta"] \arrow[ur, "\sim"'] & \end{tikzcd}$$

"counital"

**Def**: A coalgebra $(C, \Delta, \varepsilon)$ is called *cocommutative* if

$$\begin{tikzcd} C \otimes C \arrow[rr, "\tau_{C,C}"] & & C \otimes C \\ & C \arrow[ul, "\Delta"] \arrow[ur, "\Delta"'] & \end{tikzcd}$$

**Def**: Given two coalgebras $(C, \Delta, \varepsilon)$ and $(C', \Delta', \varepsilon')$, a linear map $f: C \to C'$ is a *morphism of coalgebras* if the following commute:

$$\begin{tikzcd} C \arrow[rr, "f"] \arrow[dr, "\varepsilon"'] & & C' \arrow[dl, "\varepsilon'"] \\ & \Bbbk & \end{tikzcd}$$

and

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\Delta"'] & C' \arrow[d, "\Delta'"] \\ C \otimes C \arrow[r, "f \otimes f"] & C' \otimes C' \end{tikzcd}$$

- If $C$ is a coalgebra (with coproduct $\Delta$, counit $\varepsilon$), then the *opposite coalgebra* $C^{cop}$ is the same vector space with $\Delta^{op} = \tau_{C,C} \circ \Delta$ and $\varepsilon^{op} = \varepsilon$

**Def**: A subspace $I$ of a coalgebra $(C, \Delta, \varepsilon)$ is called a *coideal* if

$$\Delta(I) \subseteq I \otimes C + C \otimes I \quad \text{and} \quad \varepsilon(I) = 0$$

**Lemma 2** (easy exercise): If $I$ is a coideal of a coalgebra $C$, then there is a unique coalgebra structure on the quotient space $C/I$ such that the canonical projection $C \to C/I$ is a coalgebra morphism.

This $C/I$ is called the *quotient coalgebra*.

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There is a simple relation between algebras and coalgebras:

> **Lemma 3**: a) The dual vector space of a coalgebra is an algebra
>
> b) The dual vector space of a finite-dimensional algebra is a coalgebra

▶ a) Dualizing $\Delta$, we get $\Delta^*:(C\otimes C)^* \longrightarrow C^*$.

On the other hand, we always have a natural map $\lambda: C^*\otimes C^* \longrightarrow (C\otimes C)^*$

Composing, we see that $(C^*, \Delta^*\circ\lambda, \varepsilon^*)$ is an algebra (check details!)

b) If $A$ is a fin. dim. algebra, then dualizing $\mu$ we get $\mu^*: A^* \to (A\otimes A)^*$ and $(A\otimes A)^* \xrightarrow[\lambda^{-1}]{\sim} A^*\otimes A^*$ given that $\dim(A)<\infty$. Hence, $(A^*, \lambda^{-1}\circ\mu^*, \eta^*)$ is a coalgebra (check details!)

Sweedler's Notations: For an element $x$ of a coalgebra $(C,\Delta,\varepsilon)$ we shall write

$$\boxed{\Delta(x) = \sum_{(x)} x'\otimes x'' \in C\otimes C}$$

Then the coassociativity reads as follows:

$$\sum_{(x)}\Big(\sum_{(x')} (x')'\otimes (x')''\Big)\otimes x'' = \sum_{(x)} x'\otimes\Big(\sum_{(x'')} (x'')'\otimes (x'')''\Big)$$

and will be rather written as

$$\boxed{\sum_{(x)} x'\otimes x''\otimes x''' \quad \text{or} \quad \sum_{(x)} x^{(1)}\otimes x^{(2)}\otimes x^{(3)}}$$

Generalizing this, we define $\Delta^{(n)}: C \longrightarrow C^{\otimes(n+1)}$ inductively via

$$\boxed{\Delta^{(1)} = \Delta\ , \quad \Delta^{(n)} = (\Delta\otimes \mathrm{id}_{C^{\otimes(n-1)}})\circ\Delta^{(n-1)} \quad \text{for } n>1}$$

Then we shall write $\boxed{\Delta^{(n)}(x) = \sum_{(x)} x^{(1)}\otimes x^{(2)}\otimes \dots \otimes x^{(n+1)}}$

On the other hand, the counitality condition reads

$$\boxed{\sum_{(x)} \varepsilon(x')\cdot x'' = x = \sum_{(x)} x'\cdot\varepsilon(x'')}$$

while the cocommutativity becomes

$$\boxed{\sum_{(x)} x'\otimes x'' = \sum_{(x)} x''\otimes x'}$$

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## Examples

1) Let $X$ be any set, set $C = \Bbbk[X] = \bigoplus_{x\in X} \Bbbk x$ with $\Delta(x) = x\otimes x$, $\varepsilon(x) = 1$ $\forall x\in X$

This is coalgebra of a set.

[ Exercise: Show that the dual algebra of $\Bbbk[X]$ is the algebra of $\Bbbk$-valued f-s on $X$.

2) Given two coalgebras $(C,\Delta,\varepsilon)$ and $(C',\Delta',\varepsilon')$, the tensor product $C\otimes C'$ is naturally equipped with coalgebra structure with

$\varepsilon_{C\otimes C'} = \varepsilon\otimes\varepsilon'$ and $\Delta_{C\otimes C'} = (id\otimes \tau_{C,C'}\otimes id)(\Delta\otimes\Delta')$

This is the tensor product of coalgebras.

[ Exercise: For any two sets $X, Y$ show that $\Bbbk[X]\otimes\Bbbk[Y] \simeq \Bbbk[X\times Y]$.

3) The dual of $A = M_n(\Bbbk) = \{ \substack{n\times n \text{ matrices} \\ \text{over } \Bbbk} \}$ is called the matrix coalgebra

[ Exercise: Let $\{x_{ij}\}_{i,j=1}^n$ be the basis dual to $\{E_{ij}\}$. Verify that
$\varepsilon(x_{ij}) = \delta_{ij}$ and $\Delta(x_{ij}) = \sum_k x_{ik}\otimes x_{kj}$.

× Bialgebras

Let us now assume that a vector space $H$ is equipped both with an algebra structure $(H,\mu,\eta)$ and a coalgebra structure $(H,\Delta,\varepsilon)$. Then, the tensor product $H\otimes H$ is equipped both with algebra and coalgebra structur.

**Proposition 1**: TFAE

(i) The maps $\mu, \eta$ are morphisms of coalgebras

(ii) The maps $\Delta, \varepsilon$ are morphisms of algebras.

‖ **Def**: A bialgebra is a quadruple $(H,\mu,\eta,\Delta,\varepsilon)$ where $(H,\mu,\eta)$ - algebra, $(H,\Delta,\varepsilon)$ - coalgebra, which satisfy the equivalent conditions of Prop 1

‖ **Def**: A morphism of bialgebras is a linear map, which is both a morphism of underlying algebras and coalgebras

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## Proof of Proposition 1

The validity of (i) is equivalent to the commutativity of 4 diagrams:

$$
\begin{tikzcd}
\Bbbk \arrow[r, "\eta"] \arrow[d, "\wr\, id"'] & H \arrow[d, "\Delta"] \\
\Bbbk\otimes\Bbbk \arrow[r, "\eta\otimes\eta"] & H\otimes H
\end{tikzcd}
,\quad
\begin{tikzcd}
\Bbbk \arrow[r, "\eta"] \arrow[dr, "id"'] & H \arrow[d, "\varepsilon"] \\
 & \Bbbk
\end{tikzcd}
,\quad
\begin{tikzcd}
H\otimes H \arrow[r, "\varepsilon\otimes\varepsilon"] \arrow[d, "\mu"'] & \Bbbk\otimes\Bbbk \arrow[d, "\mu"] \\
H \arrow[r, "\varepsilon"] & \Bbbk
\end{tikzcd}
,\quad
\begin{tikzcd}
H\otimes H \arrow[rr, "\mu"] \arrow[d, "(id\otimes\tau\otimes id)(\Delta\otimes\Delta)"'] & & H \arrow[d, "\Delta"] \\
(H\otimes H)\otimes(H\otimes H) \arrow[rr, "\mu\otimes\mu"'] & & H\otimes H
\end{tikzcd}
$$

while the validity of (ii) is equivalent to the commutativity of:

$$
\begin{tikzcd}
H\otimes H \arrow[r, "\varepsilon\otimes\varepsilon"] \arrow[d, "\mu"'] & \Bbbk\otimes\Bbbk \arrow[d, "\mu"] \\
H \arrow[r, "\varepsilon"] & \Bbbk
\end{tikzcd}
,\quad
\begin{tikzcd}
\Bbbk \arrow[r, "\eta"] \arrow[dr, "id"'] & H \arrow[d, "\varepsilon"] \\
 & \Bbbk
\end{tikzcd}
,\quad
\begin{tikzcd}
\Bbbk \arrow[r, "\eta"] \arrow[d, "id"'] & H \arrow[d, "\Delta"] \\
\Bbbk\otimes\Bbbk \arrow[r, "\eta\otimes\eta"] & H\otimes H
\end{tikzcd}
,\quad
\begin{tikzcd}
H\otimes H \arrow[rr, "\Delta\otimes\Delta"] \arrow[d, "\mu"'] & & (H\otimes H)\otimes(H\otimes H) \arrow[d, "(\mu\otimes\mu)(id\otimes\tau\otimes id)"] \\
H \arrow[rr] & & H\otimes H
\end{tikzcd}
$$

Clearly, these 4 diagrams are the same as the first four. ∎

Thus, for bialgebras we have the following simple compatibilities

$$\boxed{\varepsilon(1)=1,\quad \Delta(1)=1\otimes 1,\quad \varepsilon(xy)=\varepsilon(x)\varepsilon(y),\quad \Delta(xy)=\Delta(x)\Delta(y)}$$

**Exercise**: If $H=(H,\mu,\eta,\Delta,\varepsilon)$ is a bialgebra, then the following are also bialg-s:

$$H^{op}=(H,\mu^{op},\eta,\Delta,\varepsilon)$$
$$H^{cop}=(H,\mu,\eta,\Delta^{op},\varepsilon)$$
$$H^{op,cop}=(H,\mu^{op},\eta,\Delta^{op},\varepsilon)$$

As an immediate corollary of Lemma 3, we also get:

**Corollary 1**: A dual of a finite-dimensional bialgebra has a canonical bialgebra structure

Q: Which extra structure on a set $X$ makes $\Bbbk[X]$ into a bialgebra?
(e.g. an associative map $\mu: X\times X\to X$ with a left & right unit $e$).