# Lecture #2

Recall that given any vector space $V$, one has the <u>tensor algebra</u> of $V$, namely $T(V)=\bigoplus_{n\geq 0} V^{\otimes n}$ with the product map $V^{\otimes n}\otimes V^{\otimes m} \xrightarrow{\sim} V^{\otimes(n+m)}$. This satisfies the following universal property: $\mathrm{Hom}_{\text{ass.alg}}(T(V),A)\simeq \mathrm{Hom}_{\text{lin}}(V,A)$ $\forall$ algebra $A$

**Exercise**: Show that there is a unique bialgebra structure on $T(V)$ such that

$$\Delta(v)=v\otimes 1+1\otimes v\quad,\quad \varepsilon(v)=0\qquad \forall v\in V$$

This bialgebra is cocommutative and satisfies:

$$\begin{cases}\varepsilon(v_1\ldots v_n)=0\\ \Delta(v_1\ldots v_n)=v_1\ldots v_n\otimes 1+1\otimes v_1\ldots v_n+\sum_{p=1}^{n-1}\sum_{\substack{\sigma\in S(n)\\ \text{shuffle}}} v_{\sigma(1)}\ldots v_{\sigma(p)}\otimes v_{\sigma(p+1)}\ldots v_{\sigma(n)}\end{cases}$$

for any $v_1,\ldots,v_n\in V$ (so that $v_1\ldots v_n\in V^{\otimes n}$, where we omit $\otimes$ not to get it confused with $\otimes$ in $T(V)\otimes T(V)$)

Here, $\sigma\in S(n)$ is a $(p,n-p)$-shuffle if $\begin{cases}\sigma(1)<\sigma(2)<\ldots<\sigma(p)\\ \sigma(p+1)<\sigma(p+2)<\ldots<\sigma(n)\end{cases}$

**Def**: An element $x$ of a coalgebra $(C,\Delta,\varepsilon)$ is <u>primitive</u> if $\Delta(x)=x\otimes 1+1\otimes x$

Let $\mathrm{Prim}(C)=\{\text{all primitive elts}\}$.

**Lemma 1**: a) If $x$ - primitive elt of a bialgebra, then $\varepsilon(x)=0$

b) If $x,y$ - primitive elts of a bialgebra, then $[x,y]:=xy-yx$ - primitive

▸ a) Recall $\mathrm{Id}=(\varepsilon\otimes \mathrm{id})\Delta \Rightarrow x=(\varepsilon\otimes \mathrm{id})(x\otimes 1+1\otimes x)=\varepsilon(x)+\underbrace{\varepsilon(1)}_{=1}\cdot x\Rightarrow \varepsilon(x)=0$

b) $$\left.\begin{aligned}\Delta(xy)&=xy\otimes 1+1\otimes xy+x\otimes y+y\otimes x\\ \Delta(yx)&=yx\otimes 1+1\otimes yx+y\otimes x+x\otimes y\end{aligned}\right\}\Rightarrow \Delta([x,y])=[x,y]\otimes 1+1\otimes [x,y]$$

■

Given a bialgebra $H$ and $x_1,\ldots,x_n\in \mathrm{Prim}(H)$, consider $V=\bigoplus_{i=1}^{n} kv_i$.
By universality, $\exists!$ algebra morphism $f: T(V)\to H$ such that $f(v_i)=x_i$.

**Exercise**: Show that $f: T(V)\to H$ is a morphism of bialgebras

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Given an algebra $(A, \mu, \eta)$ and a coalgebra $(C, \Delta, \varepsilon)$, one introduces the following bilinear map "convolution" on the space of linear maps $\mathrm{Hom}_{\mathrm{en}}(C, A)$:

$$\boxed{\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[rrr, bend right, "f*g"'] & C\otimes C \arrow[r, "f\otimes g"] & A\otimes A \arrow[r, "\mu"] & A \end{tikzcd} \qquad \forall f, g \in \mathrm{Hom}_{\mathrm{en}}(C, A)}$$

Equivalently, this just means that

$$\boxed{(f*g)(x) = \sum_{(x)} f(x')\cdot g(x'')}$$

**Proposition 1**: a) $(\mathrm{Hom}(C, A), *, \eta\circ\varepsilon)$ is an algebra

b) The natural map $A\otimes C^* \xrightarrow{\bar{\jmath}} \mathrm{Hom}(C, A)$ is an algebra morphism, where $C^*$ is viewed as an algebra dual to $C$.

▷ a) The associativity of $*$ follows from associativity of $\mu$ & coass. of $\Delta$.

$$((f*g)*h)(x) = \sum_{(x)} f(x')g(x'')h(x''') = (f*(g*h))(x).$$

The unit property of $\eta\circ\varepsilon : C \xrightarrow{\varepsilon} \Bbbk \xrightarrow{\eta} A$ follows from:

$$(\eta\varepsilon * f)(x) = \sum_{(x)} \varepsilon(x')f(x'') = f\Big(\sum_{(x)} \varepsilon(x')x''\Big) = f(x) \quad \text{due to } (\varepsilon\otimes id)\Delta = id$$

$$(f*\eta\varepsilon)(x) = \sum_{(x)} f(x')\varepsilon(x'') = f\Big(\sum_{(x)} x'\varepsilon(x'')\Big) = f(x) \quad \text{due to } (id\otimes\varepsilon)\Delta = id.$$

b) To check that $\bar{\jmath}$ is compatible with products, we compute:

$$(\bar{\jmath}(a\otimes\alpha) * \bar{\jmath}(b\otimes\beta))(x) = \sum_{(x)} \alpha(x')\beta(x'')ab = (\alpha\beta)(x)\cdot ab = (\bar{\jmath}(ab\otimes\alpha\beta))(x)$$

for any $a, b\in A$, $\alpha, \beta \in C^*$, and $x\in C$.

The compatibility of units follows from:

$$(\bar{\jmath}(1\otimes\varepsilon))(x) = \varepsilon(x)1 = \eta\varepsilon(x)$$

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Rmk: When $A = \Bbbk$, the resulting algebra structure on $C^*$ is the same as in Lecture 1.

**Def**: Let $(H, \mu, \eta, \Delta, \varepsilon)$ be a bialgebra and $\star$ be the convolution on $\mathrm{End}(H) = \mathrm{Hom}_{lin}(H,H)$. An elt $S \in \mathrm{End}(H)$ is called an *antipode* if

$$\boxed{S \star id_H = id_H \star S = \eta \circ \varepsilon}$$

**Def**: A *Hopf algebra* is a bialgebra with an antipode. A morphism of Hopf algebras is a morphism of bialgebras commuting with antipodes.

Note that having an antipode is a property and not an extra data, i.e. an antipode is unique if it exists. Indeed, assume $S, S'$ – two antipodes, then:

$$S = S \star (\eta\varepsilon) = S \star (id_H \star S') = (S \star id_H) \star S' = (\eta\varepsilon) \star S' = S' \Rightarrow S = S'.$$

Notation: A Hopf algebra $H$ is the following tuple $H = (H, \mu, \eta, \Delta, \varepsilon, S)$

Rmk: The above condition on the antipode reads as:

$$\boxed{\sum_{(x)} S(x') x'' = \underbrace{\varepsilon(x) \cdot 1}_{=\eta\varepsilon(x)} = \sum_{(x)} x' S(x'')}$$

The following is easy:

**Lemma 2**: Let $H$ be a finite-dimensional Hopf algebra with antipode $S$. Then the bialgebra $H^*$ is a Hopf algebra with antipode $S^*$.

▷ For any $\alpha \in H^*$, $x \in H$, we have:

$$\left(\sum_{(\alpha)} \alpha' S^*(\alpha'')\right)(x) = \sum_{(\alpha)(x)} \alpha'(x') (S^*(\alpha''))(x'') = \sum_{(\alpha)(x)} \alpha'(x')\alpha''(S(x'')) =$$

$$= \alpha\left(\sum_{(x)} x' S(x'')\right) = \alpha(\eta\varepsilon(x)) = (\varepsilon^* \eta^*(\alpha))(x)$$

The other equality $\left(\sum_{(\alpha)} S^*(\alpha')\alpha''\right)(x) = (\varepsilon^* \eta^* \alpha)(x)$ is analogous ∎

**Q**: If $X$ is a monoid (i.e. has associative map $\mu: X \times X \to X$ with unit elt) then $\Bbbk[X]$ is a bialgebra (as was in the Q from Lecture 1). Under which conditions, does $\Bbbk[X]$ have an antipode?

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We conclude with the following result:

**Proposition 2:** Let $(H, \mu, \eta, \Delta, \varepsilon, S)$ be a Hopf algebra. Then:

a) $S$ is a bialgebra morphism from $H$ to $H^{op,cop}$, i.e.

$$\boxed{S(xy) = S(y)S(x), \quad S(1)=1, \quad (S\otimes S)\Delta = \Delta^{op} S, \quad \varepsilon S = \varepsilon}$$

b) TFAE:

i) $S^2 = id_H$

ii) $\sum_{(x)} S(x'')x' = \varepsilon(x)1 \quad \forall x \in H$

iii) $\sum_{(x)} x'' S(x') = \varepsilon(x)1 \quad \forall x \in H$

c) If $H$ is commutative or cocommutative, then $S^2 = id_H$

We shall be proving in the reverse order.

c) If $H$-commutative, then

$$\varepsilon(x)1 = \eta\varepsilon(x) = \sum_{(x)} x'S(x'') \underset{comm}{=\!=} \sum_{(x)} S(x'')x' \overset{b)}{\Longrightarrow} S^2 = id_H$$

If $H$-cocommutative, then

$$\varepsilon(x)1 = \eta\varepsilon(x) = \sum_{(x)} S(x')x'' \underset{cocomm}{=\!=} \sum_{(x)} S(x'')x' \overset{b)}{\Longrightarrow} S^2 = id_H$$

b) We shall prove i) $\Leftrightarrow$ ii), while i) $\Leftrightarrow$ iii) is completely analogous.

i) $\Rightarrow$ ii)

$$\sum_{(x)} S(x'')x' \underset{i)}{=} S^2\Big(\sum_{(x)} S(x'')x'\Big) \underset{a)}{=} S\Big(\sum_{(x)} S(x')S^2(x'')\Big) \underset{i)}{=} S\Big(\sum_{(x)} S(x')x''\Big) =$$

$$= S(\varepsilon(x)1) = \varepsilon(x)1$$

ii) $\Rightarrow$ i)

It suffices to prove $S \star S^2 = \eta\varepsilon$, due to the uniqueness of inverse, which is proved as follows:

$$(S\star S^2)(x) = \sum_{(x)} S(x')S^2(x'') \underset{a)}{=} S\Big(\sum_{(x)} S(x'')x'\Big) \underset{ii)}{=} S(\varepsilon(x)1) = \varepsilon(x)1 = \eta\varepsilon(x)$$

It thus remains to prove part a)

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(Continuation – proof of part a))

a) • The equality $S(1)=1$ follows from evaluating $id_H * S = \eta\varepsilon$ at $x=1$.

• The equality $\varepsilon S = \varepsilon$ follow from

$$\varepsilon S(x) = \varepsilon\Big(S\big(\sum_{(x)} \varepsilon(x')x''\big)\Big) = \varepsilon\Big(\sum_{(x)} \varepsilon(x')S(x'')\Big) = \varepsilon(\eta\varepsilon(x)) = \varepsilon(x)$$

• To prove $S(xy)=S(y)S(x)$, we define $\nu, \rho \in Hom_{lin}(H\otimes H, H)$ via

$$\nu(x\otimes y) = S(y)S(x) \quad \text{and} \quad \rho(x\otimes y) = S(xy).$$

It suffices to prove $\rho * \mu = \mu * \nu = \eta\varepsilon$, since $\eta\varepsilon$ is the identity w.r.t. $*$.

∘ $$(\rho*\mu)(x\otimes y) = \sum_{(x\otimes y)} \rho((x\otimes y)')\mu((x\otimes y)'') = \sum_{(x)(y)} \rho(x'\otimes y')\mu(x''\otimes y'') = \sum_{(x)(y)} S(x'y')x''y''$$

$$\overset{\Delta\text{-alg. hom}}{=\!=\!=} \sum_{(x)(y)} S((xy)')(xy)'' = \eta\varepsilon(xy)$$

∘ $$(\mu*\nu)(x\otimes y) = \sum_{(x\otimes y)} \mu((x\otimes y)')\nu((x\otimes y)'') = \sum_{(x)(y)} \mu(x'\otimes y')\,\nu(x''\otimes y'') =$$

$$= \sum_{(x)(y)} x'\underbrace{y'S(y'')}S(x'') = \sum_{(x)} x'\cdot\eta\varepsilon(y)\cdot S(x'') = \eta\varepsilon(y)\cdot\eta\varepsilon(x) = \eta\varepsilon(xy)$$

• Finally to prove $\Delta S = (S\otimes S)\Delta^{op}$, we define $\nu, \rho \in Hom_{lin}(H, H\otimes H)$ via

$$\rho = \Delta S \quad \text{and} \quad \nu = (S\otimes S)\Delta^{op}$$

Similarly to above, it suffices to check $\rho*\Delta = \Delta*\nu = (\eta\otimes\eta)\varepsilon$.

∘ $$(\rho*\Delta)(x) = \sum_{(x)} \Delta(S(x'))\Delta(x'') = \Delta\Big(\sum_{(x)} S(x')x''\Big) = \Delta(\eta\varepsilon(x)) = ((\eta\otimes\eta)\varepsilon)(x)$$

∘ $$(\Delta*\nu)(x) = \sum_{(x)} \Delta(x')\big((S\otimes S)\Delta^{op}(x'')\big) = \sum_{(x)} (x'\otimes x'')(S(x'''')\otimes S(x''')) =$$

$$= \sum_{(x)} x'S(x'''')\otimes x''S(x''') = \sum_{(x)} x'S(x''')\otimes \varepsilon(x'')1 =$$

$$= \sum_{(x)} x'\varepsilon(x'')S(x''')\otimes 1 = \sum_{(x)} x'S(x'')\otimes 1 = \varepsilon(x)1\otimes 1 = ((\eta\otimes\eta)\varepsilon)(x)$$

$\blacksquare$

**Corollary 1**: Let $H = (H, \mu, \eta, \Delta, \varepsilon, S)$ be a Hopf algebra.

a) Then $H^{op,cop} = (H, \mu^{op}, \eta, \Delta^{op}, \varepsilon, S)$ is also a Hopf algebra and $S: H \to H^{op,cop}$ is a morphism of Hopf alg-s.

b) If $S$ is an isomorphism, then $H^{op} = (H, \mu^{op}, \eta, \Delta, \varepsilon, S^{-1})$ and $H^{cop} = (H, \mu, \eta, \Delta^{op}, \varepsilon, S^{-1})$ are isom. Hopf alg-s, the isomorphism given by $S$.