Lecture # 3

Recall that a condition on the antipode reads as follows:

$$\boxed{\sum_{(x)} S(x')x'' = \varepsilon(x)1 = \sum_{(x)} x'S(x'') \quad \forall x \in H} \quad (\Diamond)$$

The following is helpful:

Lemma 1: Let H be a bialgebra and $S: H \to H^{op}$ be an algebra morphism. Assume H is generated as an algebra by a subset X and $(\Diamond)$ holds for all $x \in X$. Then, S is an antipode for H.

▷ It suffices to prove that if x, y satisfy $(\Diamond)$, then so does $x \cdot y$

$$\sum_{(xy)} (xy)' S((xy)'') = \sum_{(x)(y)} x'y' S(x''y'') = \sum_{(x)(y)} x'y' S(y'')S(x'') = \varepsilon(y)\sum_{(x)} x'S(x'') = \underbrace{\varepsilon(y)\varepsilon(x)}_{=\varepsilon(xy)}$$

and the other check is similar ◻

Example 1: Recall the bialgebra structure on $T(V)$ from Lecture 2. Then $S: T(V) \to T(V)$ given by $S(v_1 \otimes \_ \otimes v_k) = (-1)^k v_k \otimes \_ \otimes v_1$ is an antipode.

Example 2: The symmetric algebra $S(V) := T(V)/I(V)$, where $I(V)$ is the 2-sided ideal generated by $\{xy - yx \mid x, y \in V\}$ has an induced Hopf algebra structure with antipode $S(v_1 \ldots v_k) = (-1)^k v_1 \_ v_k$

(Need to check $I(V)$ is a coideal, i.e. $\varepsilon(V) = 0$ which is clear and $\Delta(I(V)) \subset I(V) \otimes T(V) + T(V) \otimes I(V)$, which follows from $\Delta([x,y]) = [x,y] \otimes 1 + 1 \otimes [x,y]$)

Def: An element $x \neq 0$ of a coalgebra (H, Δ, ε) is *group-like* if $\Delta(x) = x \otimes x$.
We denote $\mathcal{G}(H) = \{\text{all grouplike elts}\}$

Lemma 2: Let H be a bialgebra. Then $\mathcal{G}(H)$ is a monoid w.r.t. multiplication of H. Furthermore, if H is a Hopf algebra, then $x \in \mathcal{G}(H) \Rightarrow S(x) \in \mathcal{G}(H)$ and $xS(x) = 1 = S(x)x$, i.e. $\mathcal{G}(H)$ is a group.

▷ As $\Delta^{op} \circ S = (S \otimes S) \circ \Delta$, we get $\Delta(x) = x \otimes x \overset{S\otimes S}{\Longrightarrow} \Delta^{op}(S(x)) = S(x) \otimes S(x) \Rightarrow S(x) \in \mathcal{G}(H)$

$\Delta(xy) = \Delta(x) \cdot \Delta(y) \Rightarrow xy \in \mathcal{G}(H)$ if $x, y \in \mathcal{G}(H)$.

Finally, $xS(x) = \varepsilon(x)1 = S(x)x$. But $x = \mu \circ (\varepsilon \otimes \mathrm{Id}) \circ \Delta(x) = x\varepsilon(x) \overset{x \neq 0}{\Longrightarrow} \varepsilon(x) = 1 \Rightarrow xS(x) = 1$ ◻

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Bialgebra modules

- If A - bialgebra, U,V - A-modules, then we make $U\otimes V$ into A-module via

$$\boxed{A \xrightarrow{\Delta} A\otimes A \curvearrowright U\otimes V \quad \text{i.e.} \quad a(u\otimes v) = \sum_{(a)} a'u \otimes a''v}$$

- Any vector space V can be equipped with a trivial A-module structure:

$$\boxed{a(v) = \varepsilon(a)\cdot v \quad \forall a\in A,\, v\in V}$$

The following is easy:

Lemma 3: If A - bialgebra, U,V,W - A-modules, k - trivial A-module, then $(U\otimes V)\otimes W \simeq U\otimes(V\otimes W)$, $k\otimes V \simeq V \simeq V\otimes k$ as A-modules.
Moreover, if A is cocommutative, then $\tau_{V,W}: V\otimes W \xrightarrow{\sim} W\otimes V$ - A-module isomorphism

Now if A is actually a Hopf algebra, then given A-modules V, U, the space $\mathrm{Hom}_{lin}(V,U)$ of linear maps $V\to U$ is equipped with A-module structure:

$$\boxed{\begin{array}{l} A \xrightarrow{\Delta} A\otimes A \xrightarrow{1\otimes S} A\otimes A^{op} \curvearrowright \mathrm{Hom}(V,U) \\ \qquad\qquad\qquad ((a\otimes b)f)(v) := a(f(bv)) \end{array}}$$

so that in Sweedler's notation we get

$$\boxed{(af)(v) = \sum_{(a)} a' f(S(a'')v)}$$

Dual module

Applying the above construction for specific case $U = k$ - trivial module, we get an A-module structure on the dual space V^*. Explicitly:

$$(af)(v) = \sum_{(a)} \varepsilon(a') f(S(a'')v) = f\Big(S\Big(\sum_{(a)}\varepsilon(a')a''\Big)v\Big) = f(S(a)v), \text{ so that}$$

$$\boxed{(af)(v) = f(S(a)v)}$$

! Skipped Prop III.5.2, III.5.3 from Kassel's book → you are welcome/suggested to read those
↳ if we need them latter, then we'll discuss

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Comodules over coalgebras

Similarly to how coalgebras were "dual" (reversing arrows) to algebras, the notion of algebra modules admits a dual notion of coalgebra comodules.

Recall: If A - assoc. algebra, then an A-module is a pair (M, μ_M), where M is a v. space, μ_M ("action") $: A \otimes M \to M$ - linear map s.t.

$$\begin{tikzcd} A\otimes A\otimes M \arrow[r, "\mu\otimes id"] \arrow[d, "id\otimes\mu_M"'] & A\otimes M \arrow[d, "\mu_M"] \\ A\otimes M \arrow[r, "\mu_M"] & M \end{tikzcd} \quad \& \quad \begin{tikzcd} \Bbbk\otimes M \arrow[r, "\eta\otimes id"] \arrow[dr, equal] & A\otimes M \arrow[d, "\mu_M"] \\ & M \end{tikzcd}$$

Thus, reversing arrows we get:

Def: Let (C, Δ, ε) be a coalgebra. A left *C-comodule* is a pair (N, Δ_N), where N - v. space, $\Delta_N : N \to C \otimes N$ - linear map ("coaction") s.t.

$$\begin{tikzcd} N \arrow[r, "\Delta_N"] \arrow[d, "\Delta_N"'] & C\otimes N \arrow[d, "\Delta\otimes id"] \\ C\otimes N \arrow[r, "id\otimes\Delta_N"] & C\otimes C\otimes N \end{tikzcd} \quad \& \quad \begin{tikzcd} \Bbbk\otimes N & C\otimes N \arrow[l, "\varepsilon\otimes id"'] \\ & N \arrow[u, "\Delta_N"'] \arrow[ul, equal] \end{tikzcd}$$

Def: Given a coalgebra C and two C-comodules $(N, \Delta_N), (N', \Delta_{N'})$, a linear map $f : N \to N'$ is called a *morphism* of comodules if

$$\begin{tikzcd} N \arrow[r, "f"] \arrow[d, "\Delta_N"'] & N' \arrow[d, "\Delta_{N'}"] \\ C\otimes N \arrow[r, "id\otimes f"] & C\otimes N' \end{tikzcd}$$

Def: A subspace $N' \subseteq N$ is a *subcomodule* if $\Delta_N(N') \subset C \otimes N'$.

Rmk: Likewise, one can also define right C-comodules as above, but now with $\Delta_N : N \to N \otimes C$. However, it is an easy exercise to see that right C-comodule is the same as a left C^{op}-comodule.

Example 1: Any coalgebra (C, Δ, ε) is a comodule over itself with $\Delta_C = \Delta$

The following result shows that modules and comodules are indeed dual in the same sense algebras and coalgebras are dual

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Lemma 4: (a) If (N, Δ_N) is a comodule over coalgebra (C, Δ, ε), then N^* is a right C^*-module via $N^* \otimes C^* \xrightarrow[\text{lin. map}]{\text{natural}} (C \otimes N)^* \xrightarrow[\Delta_N^*]{} N^*$

(b) If (M, μ_M) is a right module over f.dim. algebra (A, μ, η), then M^* is a left A^*-comodule via $M^* \xrightarrow[\mu_M^*]{} (M \otimes A)^* \xrightarrow[\text{natural isom.}]{\simeq} A^* \otimes M^*$

Exercise: Prove the above Lemma.

We shall use the Sweedler's notations for comodules (N, Δ_N), i.e.

$$\boxed{\Delta_N(x) = \sum_{(x)} x_C \otimes x_N} \quad (\text{with } x_C \in C,\ x_N \in N)$$

so that

$$\boxed{\sum_{(x)} \varepsilon(x_C) \cdot x_N = x \quad , \quad \sum_{(x)} (x_C)' \otimes (x_C)'' \otimes x_N = \sum_{(x)} x_C \otimes (x_N)_C \otimes (x_N)_N}$$

suppress (x_C) suppress (x_N)

Example 2 (tensor product construction): Let $(H, \mu, \eta, \Delta, \varepsilon)$ be a bialgebra. If M, N – H-comodules, then $M \otimes N$ has a natural H-comodule structure via

$$\boxed{\Delta_{M\otimes N} : M \otimes N \xrightarrow{\Delta_M \otimes \Delta_N} H \otimes M \otimes H \otimes N \xrightarrow{id \otimes \tau \otimes id} H \otimes H \otimes M \otimes N \xrightarrow{\mu \otimes id \otimes id} H \otimes M \otimes N}$$

- Indeed, the commutativity of the diagram

$$\begin{tikzcd} M \otimes N \arrow[r, "\Delta_{M\otimes N}"] \arrow[dr, equal] & H \otimes M \otimes N \arrow[d, "\varepsilon \otimes id_{M\otimes N}"] \\ & \Bbbk \otimes M \otimes N \end{tikzcd}$$

follows from such diagrams for M, N, and $\varepsilon(xy) = \varepsilon(x)\varepsilon(y)$

- Likewise, the commutativity of

$$\begin{tikzcd} M \otimes N \arrow[r, "\Delta_{M\otimes N}"] \arrow[d, "\Delta_{M\otimes N}"'] & H \otimes M \otimes N \arrow[d, "id \otimes \Delta_{M\otimes N}"] \\ H \otimes M \otimes N \arrow[r, "\Delta \otimes id \otimes id"] & H \otimes H \otimes M \otimes N \end{tikzcd}$$

follows from such diagrams for M, N and $\Delta(xy) = \Delta(x)\Delta(y)$

Example 3 (trivial comodule): Given a bialgebra $(H, \mu, \eta, \Delta, \varepsilon)$ any v. space V is endowed with a trivial comodule structure via $\Delta_V : V \simeq \Bbbk \otimes V \xrightarrow{\eta \otimes id} H \otimes V$.

Example 4 (free comodule): Given a coalgebra (C, Δ, ε) and any v. space V the free C-comodule on V is $(C \otimes V, \Delta \otimes id)$

Exercise: Carefully check the above 3 examples

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The following result is left as a simple exercise (cf. Lemma 3):

Lemma 5: (a) If H - bialgebra, M, N, P - H-comodules, $\Bbbk$ - trivial comodule, then $(M\otimes N)\otimes P \simeq M\otimes(N\otimes P)$, $\Bbbk\otimes M \simeq M \simeq M\otimes \Bbbk$ as H-comod.

(b) If H - cocommutative, then $\tau : M\otimes N \xrightarrow{\sim} N\otimes M$ is H-comod. isom.

Bimodules

Let H - bialgebra and a v. space M be equipped with both module & comodule structures over H, i.e. have the corresponding $\mu_M : H\otimes M \longrightarrow M$, $\Delta_M : M \longrightarrow H\otimes M$.
Note that $H\otimes M$ can be also viewed as a module or comodule over H.

Proposition 1: TFAE

a) μ_M - morphism of comodules

b) Δ_M - morphism of modules

▶ a) $\Leftrightarrow$ commutativity of

$$\begin{array}{ccc} H\otimes M & \xrightarrow{\Delta_{H\otimes M}} & H\otimes H\otimes M \\ \downarrow \mu_M & & \downarrow id\otimes\mu_M \\ M & \xrightarrow[\Delta_M]{} & H\otimes M \end{array}$$

b) $\Leftrightarrow$ commutativity of

$$\begin{array}{ccc} H\otimes M & \xrightarrow{id\otimes\Delta_M} & H\otimes H\otimes M \\ \downarrow \mu_M & & \downarrow \mu_{H\otimes M} \\ M & \xrightarrow[\Delta_M]{} & H\otimes M \end{array}$$

Thus: a) $\Leftrightarrow$ b) follows from the equality $(id\otimes\mu_M)\circ\Delta_{H\otimes M} = \mu_{H\otimes M}\circ(id\otimes\Delta_M)$
To check the latter, let $x\in H, a\in M$ - any elts. Then:

$$(id\otimes\mu_M)\Delta_{H\otimes M}(x\otimes a) = \sum_{(x)(a)} x' a_H \otimes \mu_M(x'', a_M) = \mu_{H\otimes M}(id\otimes\Delta_M)(x\otimes a)$$ ▨

Def: Given a bialgebra H, a v. space M equipped with both module & comodule structures over H, we say M is *H-bimodule* if the equivalent conditions of Prop 1 hold.

Example (exercise): If H - bialgebra, V - v. space, then $H\otimes V$ is endowed with both $\underbrace{\text{free module structure}}_{\text{induced by } \mu}$ & $\underbrace{\text{free comodule structure}}_{\text{induced by } \Delta}$. Verify that $H\otimes V$ is an H-bimodule