Lecture #4

• Last time: modules, comodules, bimodules

Example (exercise to check): Let H be any bialgebra and V any vector space (over the same field). Then, $H\otimes V$ is endowed with both a (free) H-comodule structure as well as a natural H-module structure $H\otimes(H\otimes V)\longrightarrow H\otimes V$ via $a\otimes(b\otimes v)\mapsto ab\otimes v$.
In fact, $H\otimes V$ is an H-bimodule.

With this example in mind, we can now state the "Structure theorem for bimodules" (it will be on HWK #1):

Theorem: Let H – Hopf algebra, M – H-bimodule. Define $N:=\{m\in M \mid \Delta_M(m)=1\otimes m\}$.
Then, the multiplication map $\mu_M: H\otimes N\longrightarrow M$ is an H-bimodule isomorphism.

• General Picture

We shall now discuss how Hopf algebras and comodules over them arise naturally in the context of algebraic groups and their action on algebraic varieties.

[1] Assume an algebraic variety G is endowed with a monoid structure with unit, i.e. we have an associative map $\mu: G\times G\to G$ as well as $\eta:\{e\}\longrightarrow G$ s.t. $\mu(e,g)=g=\mu(g,e)$ $\forall g\in G$. Then, on the level of regular/polynomial functions we get the maps opposite way

$$\boxed{\Delta:=\mu^*: \mathbb{k}[G]\longrightarrow \underbrace{\mathbb{k}[G\times G]}_{=\mathbb{k}[G]\otimes\mathbb{k}[G]} \quad\text{and}\quad \varepsilon:=\eta^*: \mathbb{k}[G]\longrightarrow \mathbb{k}[\{e\}]=\mathbb{k}}$$

Remarks (physics audience): a) "Baby version" of this is G–finite monoid, $\mathbb{k}[G]=\{\mathbb{k}\text{-valued f-s on } G\}$

b) In general, given a map $\varphi: X\longrightarrow Y$ of two sets, we always get a map $\varphi^*: \mathbb{k}[Y]\longrightarrow\mathbb{k}[X]$ defined via $(\varphi^*f)(x):=f(\varphi(x))$ $\forall x\in X, f\in\mathbb{k}[Y]$.

Warning: This $\mathbb{k}[G]$ is different from the same notation we used in Lecture 1.

Exercise 1a: Verify that $(\mathbb{k}[G],\Delta,\varepsilon)$ is a coalgebra.

On top of that, $\mathbb{k}[G]$ has an obvious structure of a commutative algebra.

Exercise 1b: Verify that $\mathbb{k}[G]$ is actually a bialgebra.

2. If G is not just a monoid, but is a group, i.e. we have the inverse map $\imath: G \to G$ s.t. $G \xrightarrow{\text{diag}} G\times G \xrightarrow{id \times \imath} G\times G \xrightarrow{\mu} G$ is just sending $\forall g \mapsto e$, then the induced map $\boxed{S := \imath^* : \Bbbk[G] \to \Bbbk[G]}$ fits into the commutative diagram

$$\begin{tikzcd} \Bbbk[G] & \Bbbk[G]\otimes\Bbbk[G] = \Bbbk[G\times G] \arrow[l, "\mu"'] & \Bbbk[G]\otimes\Bbbk[G] \arrow[l, "id\times S"'] & \Bbbk[G] \arrow[l, "\Delta"'] \arrow[lll, bend left, "\eta\circ\varepsilon"] \end{tikzcd}$$

[Exercise 1c: Check the above and conclude that $\Bbbk[G]$ is a Hopf algebra if G-group

3. Finally, let's now assume that our monoid/group G algebraically acts on an algebraic variety X, denoted $G \curvearrowright X$. That is, we have a map $G\times X \xrightarrow{\mu_X} X$ which fits into

$$\begin{tikzcd} G\times G\times X \arrow[r, "id\times\mu_X"] \arrow[d, "\mu\times id"'] & G\times X \arrow[d, "\mu_X"] \\ G\times X \arrow[r, "\mu_X"] & X \end{tikzcd}$$

(and compatible with unit $e\in G$, i.e. $\mu_X(e,x)=x\ \forall x$)

then on the level of functions we get

$$\boxed{\Delta_X := \mu_X^* : \Bbbk[X] \longrightarrow \Bbbk[G\times X] = \Bbbk[G]\otimes\Bbbk[X]}$$

so that the above commutative diagram immediately implies the following:

[Exercise 1d): $\Bbbk[X]$ is a $\Bbbk[G]$-comodule via the above Δ_X

In fact, since $\Bbbk[X]$ has an internal algebra structure, one can upgrade the above by showing that (the definition is given on the next page):

[Exercise 1e): $\Bbbk[X]$ is a $\Bbbk[G]$-comodule-algebra

Remark: a) Again a "baby version" is finite monoid/group G acting on a finite set X so that $\Bbbk[X] = \{$all $\Bbbk$-valued functions on $X\}$.

b) If one considers other setups with different classes of functions, it may happen that $\Bbbk[X\times Y] \neq \Bbbk[X]\otimes\Bbbk[Y]$. In this case, a deeper analysis is needed.

Upshot: Alg. Group $G \rightsquigarrow$ Hopf algebra $\Bbbk[G]$

Alg. Gp. $G \curvearrowright$ Alg. var. $X \rightsquigarrow$ $\Bbbk[G]$-comodule-algebra $\Bbbk[X]$

Def: Let $H=(H, \mu_H, \eta_H, \Delta_H, \varepsilon_H)$ be a bialgebra, $A=(A, \mu_A, \eta_A)$ - algebra.
Then, A is called an <u>H-comodule - algebra</u> if

a) there is an H-comodule structure $\Delta_A : A \longrightarrow H \otimes A$

b) the maps $\mu_A : A \otimes A \longrightarrow A$, $\eta_A : \Bbbk \longrightarrow A$ are H-comodule morphisms.

Lemma 1: Given H, A as above with $\Delta_A : A \longrightarrow H \otimes A$ defining comodule structure, condition b) is equivalent to

b') the map $\Delta_A : A \longrightarrow H \otimes A$ is an algebra morphism

Rmk: With this result in mind, you can easily derive Exercise 1e).

▸ The proof is straightforward comparison of the commuting diagrams.

b) μ_A - H-comodule morphism iff the following commutes

$$\begin{tikzcd} A \otimes A \arrow[r, "\mu_A"] \arrow[d, "(\mu_H \otimes id \otimes id)(id \otimes \tau_{23} \otimes id)(\Delta_A \otimes \Delta_A)"'] & A \arrow[d, "\Delta_A"] \\ H \otimes (A \otimes A) \arrow[r, "id \otimes \mu_A"] & H \otimes A \end{tikzcd}$$

Fig 1

η_A - H-comodule morphism iff the following commutes

$$\begin{tikzcd} \Bbbk \arrow[r, "\eta_A"] \arrow[d, "\wr"'] & A \arrow[d, "\Delta_A"] \\ \Bbbk \otimes \Bbbk \arrow[r, "\eta_H \otimes \eta_A"] & H \otimes A \end{tikzcd}$$

Fig 2

b') Δ_A - algebra morphism iff the following commutes

$$\begin{tikzcd} A \otimes A \arrow[r, "\Delta_A \otimes \Delta_A"] \arrow[d, "\mu_A"'] & (H \otimes A) \otimes (H \otimes A) \arrow[d, "(\mu_H \otimes \mu_A)(id \otimes \tau_{23} \otimes id)"] \\ A \arrow[r, "\Delta_A"] & H \otimes A \end{tikzcd}$$

Fig 1'

unit η_A compatibility iff the following commutes

$$\begin{tikzcd} \Bbbk \arrow[r, "\sim"] \arrow[d, "\eta_A"'] & \Bbbk \otimes \Bbbk \arrow[d, "\eta_H \otimes \eta_A"] \\ A \arrow[r, "\Delta_A"] & H \otimes A \end{tikzcd}$$

Fig 2'

Clearly, Fig 1 & Fig 1' are equivalent (commutativity of), and so are Fig 2 & Fig 2' ∎

Lecture # 4

While we did discuss the very general picture of $G \curvearrowright X$ in pages 1-2, this week we shall be looking at the very simple case of this:

- $G = \mathrm{Mat}_{2\times 2}(\mathbb{k}) =: M_2(\mathbb{k})$ or $G =$ (invertible 2×2 matrices) $= GL_2(\mathbb{k})$ or $G =$ (det=1 2×2 matrices) $= SL_2(\mathbb{k})$.
- $X = \mathbb{k}^2 =$ (column vectors of height 2), so that $G \curvearrowright X$ via multiplication.

Following Kassel's notations, let $M(2) := \mathbb{k}[M_2(\mathbb{k})] = \mathbb{k}[a,b,c,d]$. Note that for any commutative algebra A, we have

$$\boxed{\begin{array}{ccc} \mathrm{Hom}_{alg}(M(2), A) & \overset{1\text{-to-}1}{\longleftrightarrow} & M_2(A) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ f & \longmapsto & \begin{pmatrix} f(a) & f(b) \\ f(c) & f(d) \end{pmatrix} \end{array}}$$

The general scheme from p.1 shows that multiplication of 2×2 matrices induces a coproduct map

$$\boxed{\Delta : M(2) \longrightarrow M(2)\otimes M(2) \text{ given by } \Delta\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}}$$

i.e. $\Delta(a) = a\otimes a + b\otimes c$, $\Delta(b) = a\otimes b + b\otimes d$, $\Delta(c) = c\otimes a + d\otimes c$, $\Delta(d) = c\otimes b + d\otimes d$.

Likewise, an embedding of the unit matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \hookrightarrow M_2(\mathbb{k})$ induces a counit map

$$\boxed{\varepsilon : M(2) \longrightarrow \mathbb{k} \quad \text{given by} \quad \varepsilon\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$$

<u>Thus</u>: $M(2)$ is a bialgebra (as follows from Exercise 1b))

However, to get a Hopf algebra that way, one would need the inverse map on matrices. To get this, we shall restrict to invertible or (det=1) matrices. Following Kassel (Sect I.5), define

$$\boxed{\begin{aligned} GL(2) &:= \mathbb{k}[GL_2(\mathbb{k})] = M(2)[t]/((ad-bc)t-1) \\ SL(2) &:= \mathbb{k}[SL_2(\mathbb{k})] = GL(2)/(t-1). \end{aligned}}$$

so that for any commutative algebra A we have

$$\boxed{\mathrm{Hom}_{alg}(GL(2), A) = GL_2(A) \quad \text{and} \quad \mathrm{Hom}_{alg}(SL(2), A) = SL_2(A)}$$

<u>Then</u>: inversion of matrices in $GL_2(\mathbb{k})$, $SL_2(\mathbb{k})$ will make $GL(2)$, $SL(2)$ into Hopf algebras.