Lecture #5

Last time we introduced algebras $M(2)$, $GL(2)$, $SL(2)$ as algebras of regular functions on $Mat_{2\times 2}(\mathbb{k})$, $GL_2(\mathbb{k})$, $SL_2(\mathbb{k})$, respectively. We also concluded:

- $M(2)$ - bialgebra
- $GL(2)$, $SL(2)$ - Hopf algebras
  (explicitly: $S\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (ad-bc)^{-1}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = t\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, $S(t)=t^{-1}$)

Finally, following the general picture from Lecture 4, we see that the usual multiplication $Mat_{2\times 2}(\mathbb{k}) \times \mathbb{k}^2 \longrightarrow \mathbb{k}^2$ of 2×2 matrices by column vectors of height 2 gives rise to:

Lemma 1: The algebra $A = \mathbb{k}[\mathbb{k}^2] = \mathbb{k}[x,y]$ has a comodule-algebra structure over $M(2)$, $GL(2)$, $SL(2)$ given explicitly via

$$\Delta_A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} x \\ y \end{pmatrix}, \text{ i.e. } \begin{array}{l} \Delta_A(x) = a\otimes x + b\otimes y \\ \Delta_A(y) = c\otimes x + d\otimes y \end{array}$$

Rmk: Note that each homogeneous piece $\mathbb{k}[x,y]_n = \{\text{degree } n \text{ pol-s in } x,y\}$ is a subcomodule and all $\mathbb{k}[x,y]$ is a direct sum of these finite dimensional subcomodules.

For the rest of today, we will be interested in the q-deformation of above. Here q is an invertible (i.e. nonzero) elt of $\mathbb{k}$. We shall introduce both q-analogues of $\mathbb{k}[x,y]$ as well as $M(2)$, $GL(2)$, $SL(2)$ so that all structures as above are still present.

Def: a) The quantum plane is an algebra $\mathbb{k}_q[x,y] := \mathbb{k}\langle x,y\rangle/(yx - qxy)$

b) Given any (non-commutative) algebra R, an R-point of the quantum plane is a pair $(X,Y)\in R^2$ s.t. $YX = qXY$

- Note that $\mathbb{k}_q[x,y]$ is still $\mathbb{N}$-graded via $\deg x = 1 = \deg y$
- Part b) just says $Hom_{alg}(\mathbb{k}_q[x,y], R) \xleftrightarrow{1\text{-to-}1} \{(X,Y)\in R^2 \mid YX = qXY\}$
- Finally, for $q=1$ we see that $\mathbb{k}_{q=1}[x,y] = \mathbb{k}[x,y]$

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Now we would like to introduce a q-version of $M(2) = \Bbbk[a,b,c,d]$, which was determined by pairwise commutativity of a,b,c,d.

So: we need some set of rel-s on a,b,c,d depending on q s.t. at $q=1$ they would give us back pairwise commutativity.

Idea: Think of $Mat_{2\times 2}(\Bbbk)$ as symmetries of $\Bbbk^2$ = column vectors of height 2.

To this end we assume $q^2 \neq -1$, consider q-commuting variables x, y (i.e. $yx = qxy$) and 4 variables a,b,c,d that commute with x & y. Define x', y', x'', y'' via

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \qquad (x''\ y'') = (x\ y)\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Proposition 1: TFAE:

(1) $y'x' = qx'y'$, $y''x'' = qx''y''$

(2) $ba = qab$, $ca = qac$, $db = qbd$, $dc = qcd$, $bc = cb$, $ad - da = (q^{-1} - q)bc$

The implication (2) ⇒ (1) is straightforward, so we will only prove 1) ⇒ 2).

- $y'x' = qx'y' \Leftrightarrow (cx+dy)(ax+by) = q(ax+by)(cx+dy)$

$$\Updownarrow$$

$$x^2 \cdot ca + y^2 \cdot db + xy(cb + qda) = x^2 \cdot qac + y^2 \cdot qbd + xy(qad + q^2 bc)$$

So: $\boxed{ca = qac, \ db = qbd, \ ad - da = q^{-1}cb - qbc}$

- $y''x'' = qx''y'' \overset{\text{similarly}}{\Longrightarrow} \boxed{ba = qab, \ dc = qcd, \ ad - da = q^{-1}bc - qcb}$

Comparing two f-las for $ad - da$, we get $(q + q^{-1})(bc - cb) = 0 \overset{q^2 \neq -1}{\Longrightarrow} \boxed{bc = cb}$

Hence, also $\boxed{ad - da = (q^{-1} - q)bc}$ ∎

Def: The algebra $M_q(2)$ is the quotient algebra of the free algebra in a,b,c,d by six rel-s of Prop 1(2).

Note that for any algebra R, we have a bijection

$$\mathrm{Hom}_{alg}(M_q(2), R) \overset{1\text{-to-}1}{\longleftrightarrow} \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Mat_{2\times 2}(R) \,\middle|\, \begin{matrix} BA = qAB, CA = qAC, DB = qBD \\ DC = qCD, BC = CB, AD - DA = (q^{-1} - q)BC \end{matrix} \right\}$$

Such a quadruple $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is called an *R-point of $M_q(2)$*

Rem: a) $M_q(2)$ is $\mathbb{N}$-graded via $\deg(a,b,c,d)=1$

b) $M_{q=1}(2)=M(2)$

c) If $q^2\neq -1$, R-algebra, $\begin{pmatrix} A & B \\ C & D \end{pmatrix}\in \mathrm{Mat}_{2\times 2}(R)$ is an R-point of $M_q(2)$ iff $\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} X \\ y \end{pmatrix}$ and $\begin{pmatrix} A & C \\ B & D \end{pmatrix}\begin{pmatrix} X \\ y \end{pmatrix}$ are R'-points of $M_q(2)$, where

$$R' = R\langle X,y\rangle/(yX-qXy).$$

Evoking how $GL(2)$, $SL(2)$ were introduced it becomes clear we need a q-analogue of $\det = ad-bc$.

Def: Define $\det_q \in M_q(2)$ ("quantum determinant") via $\det_q = ad-q^{-1}bc = da-qbc$

Lemma 2: $\det_q$ is a central element of $M_q(2)$

$a\cdot \det_q = a(da-qbc) = ada - q\cdot abc = ada - q\cdot q^{-1}\, bac = ada - q\cdot q^{-1} q^{-1}\, bca = \underbrace{(ad-q^{-1}bc)}_{\det_q}a$

Commutativity of $\det_q$ with b,c,d is checked similarly ∎

By analogy with the previous conventions, given an R-point $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ of $M_q(2)$, its quantum determinant is $\mathrm{Det}_q\begin{pmatrix} A & B \\ C & D \end{pmatrix} := AD-q^{-1}BC = DA-qBC$.

Proposition 2: Let R be an algebra, $m=\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, $m'=\begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}$ be two R-points of $M_q(2)$ s.t. $\{A,B,C,D\}$ commute with $\{A',B',C',D'\}$. Then:

a) $m'' := m'\cdot m$ is also an R-point of $M_q(2)$

b) $\mathrm{Det}_q(m'') = \mathrm{Det}_q(m')\cdot \mathrm{Det}_q(m)$

c) $\begin{pmatrix} D & -qB \\ -q^{-1}C & A \end{pmatrix}$ is an R-point of $M_{q^{-1}}(2)$, as well as an R^{op}-point of $M_q(2)$

- We shall crucially use this property a) next time to endow $M_q(2)$ with a bialgebra structure, as well as use b) to introduce $GL_q(2)$, $SL_q(2)$, and finally use c) to equip the latter two with Hopf alg. structures.
- A conceptual proof of part b) is in [Homework #1, Exercise 6]

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Proof of Proposition 2

a) Can be checked by direct computations. But to prove it faster, consider $R' := R\langle x,y\rangle/(yx-qxy)$. Define $\begin{pmatrix} A'' & B'' \\ C'' & D'' \end{pmatrix} = m'' = m'm = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix}$

Then: m'' is an R-point of $M_q(2)$ iff $m''\cdot\begin{pmatrix} x \\ y\end{pmatrix}$ and $(m'')^t\begin{pmatrix} x \\ y\end{pmatrix}$ are R'-points of the quantum plane.

But: $m\cdot\begin{pmatrix} x \\ y\end{pmatrix}$ are R'-points of q. plane $\Rightarrow \underbrace{m'\cdot\left(m\cdot\begin{pmatrix} x \\ y\end{pmatrix}\right)}_{=m''\cdot\begin{pmatrix} x \\ y\end{pmatrix}}$ – R'-points of q. plane

Similarly for $(m'')^t\begin{pmatrix} x \\ y\end{pmatrix}$

b) $\mathrm{Det}_q(m'') = (A'A+B'C)(C'B+D'D) - q^{-1}(A'B+B'D)(C'A+D'C) \overset{A,B,C,D \text{ comm. with } A',B',C',D'}{=\!=\!=}$

$= A'C'AB + A'D'AD + B'C'CB + B'D'CD - q^{-1}A'C'BA - q^{-1}A'D'BC - q^{-1}B'C'DA - q^{-1}B'D'DC$

$= (A'D'AD - q^{-1}A'D'BC) + (B'C'CB - q^{-1}B'C'DA) +$

$+ \underbrace{(A'C'AB - q^{-1}A'C'BA)}_{=0 \text{ as } AB=q^{-1}BA} + \underbrace{(B'D'CD - q^{-1}B'D'DC)}_{=0 \text{ as } CD=q^{-1}DC}$

$\mathrm{Det}_q(m')\mathrm{Det}_q(m) = (A'D'-q^{-1}B'C')(AD-q^{-1}BC) = A'D'AD - q^{-1}A'D'BC$
$-q^{-1}B'C'AD + q^{-2}B'C'BC$

Hence: $\mathrm{Det}_q(m'') = \mathrm{Det}_q(m')\mathrm{Det}_q(m)$ due to equality $CB - q^{-1}DA = q^{-2}BC - q^{-1}AD$ which follows from $\begin{cases} AD-DA = (q^{-1}-q)BC \\ BC = CB \end{cases}$

c) Straightforward comparison of two tuples of 6 eq-s ∎