Lecture #6

• Bialgebra structure on $M_q(2)$

Recall the algebra $M_q(2)$ introduced last time. In fact, it is a bialgebra

Proposition 1: (a) There exist algebra homomorphisms

$$\Delta: M_q(2) \longrightarrow M_q(2)\otimes M_q(2) \quad \text{given by} \quad \Delta\begin{pmatrix} a & b \\ c & d\end{pmatrix} = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d\end{pmatrix}$$

$$\varepsilon: M_q(2) \longrightarrow \Bbbk \quad \text{given by} \quad \varepsilon\begin{pmatrix} a & b \\ c & d\end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}$$

which make $M_q(2)$ into a bialgebra.

(b) $\Delta(\det_q) = \det_q \otimes \det_q$, $\quad \varepsilon(\det_q) = 1$

(a) The RHS of $\Delta\begin{pmatrix} a & b \\ c & d\end{pmatrix}$ is an $M_q(2)\otimes M_q(2)$-point of $M_q(2)$ by Proposition 2a) of Lect 5. Hence, Δ indeed defines an algebra homom. $M_q(2) \to M_q(2)\otimes M_q(2)$.

Likewise, ε defines an alg. homom. $M_q(2)\to \Bbbk$ as $\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}$ is a $\Bbbk$-point of $M_q(2)$.

Finally, coassociativity & counit axioms are checked as in $q=1$ classical case.

(b) Again, $\Delta(\det_q) = \det_q \otimes \det_q$ follows immediately from Prop 2b) of Lect 5.

Similarly, $\varepsilon(\det_q) = \varepsilon(a)\varepsilon(d) - q^{-1}\varepsilon(b)\varepsilon(c) = 1$ ∎

Rmk: a) The coalgebra structure on $M_q(2)$ is define same way as on $M(2)$

b) $M_q(2)$ is neither commutative nor cocommutative.

However, similarly to $q=1$ case of $M(2)$, bialgebra $M_q(2)$ has no antipode!

To fix this, we shall now define q-versions of $GL(2)$ and $SL(2)$.

Def: $GL_q(2) := M_q(2)[t]/(t\cdot \det_q - 1)$

$SL_q(2) := M_q(2)/(\det_q - 1) = GL_q(2)/(t-1)$

In particular, we have for any algebra R:

$$\mathrm{Hom}_{alg}(GL_q(2), R) \xleftrightarrow{1-1} \left\{ \begin{pmatrix} A & B \\ C & D\end{pmatrix} \in \mathrm{Mat}_{2\times 2}(R) \;\middle|\; \begin{array}{l} 6 \text{ rel-s} \\ + \\ AD - q^{-1}BC =: \mathrm{Det}_q \text{ is invertible}\end{array} \right\}$$

$$\mathrm{Hom}_{alg}(SL_q(2), R) \xleftrightarrow{1-1} \left\{ \begin{pmatrix} A & B \\ C & D\end{pmatrix} \in \mathrm{Mat}_{2\times 2}(R) \;\middle|\; \begin{array}{l} 6 \text{ rel-s} \\ + \\ \mathrm{Det}_q = 1\end{array} \right\}$$

The above shall be referred to as R-points of $GL_q(2)$ or $SL_q(2)$.

• Hopf algebras $GL_q(2)$ and $SL_q(2)$

We shall now q-deform Hopf algebras $GL(2)$ & $SL(2)$.

Lecture #6

Theorem 1: a) Formulas Δ, ε from Prop 1 together with
$$\Delta(t) = t\otimes t, \quad \varepsilon(t)=1$$
define bialgebra structures on $GL_q(2)$ and $SL_q(2)$.

b) Moreover, both $GL_q(2)$ and $SL_q(2)$ are Hopf alg-s with antipode
$$S\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \underbrace{\det_q^{-1}}_{=t} \cdot \begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix}$$

▶ a) It suffices to check that $(t\det_q - 1)$ is a coideal in $M_q(2)[t]$
$(\det_q - 1)$ — " — $M_q(2)$

$$\Delta(t\det_q - 1) = t\det_q \otimes t\det_q - 1\otimes 1 = (t\det_q - 1)\otimes t\det_q + 1\otimes(t\det_q - 1)$$
$$\Delta(\det_q - 1) = \det_q \otimes \det_q - 1\otimes 1 = (\det_q - 1)\otimes \det_q + 1\otimes(\det_q - 1)$$
$$\varepsilon(t\det_q - 1) = 1\cdot 1 - 1 = 0, \qquad \varepsilon(\det_q - 1) = 0$$

b) It remains to show that S is an antipode. First, let us check that S defines an algebra homomorphism $GL_q(2) \to GL_q(2)^{op}$, $SL_q(2) \to SL_q(2)^{op}$.

1. First, using Prop 2c) from Lecture 5, we see that:
$$\tilde{S}: \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix}$$
indeed defines such algebra homom. $M_q(2) \to M_q(2)^{op}$

We further set $\tilde{S}(t) := t$ to get $\tilde{S}: M_q(2)[t] \to M_q(2)[t]^{op}$. This in turn descends to $GL_q(2) \to GL_q(2)^{op}$, $SL_q(2) \to SL_q(2)^{op}$ due to:
$$\tilde{S}(t\det_q - 1) = (\tilde{S}(d)\tilde{S}(a) - q^{-1}\tilde{S}(c)\tilde{S}(b))t - 1 = (ad - q^{-1}\underbrace{cb}_{=bc})t - 1 = t\det_q - 1$$
$$\tilde{S}(\det_q - 1) \overset{\text{likewise}}{=} \det_q - 1.$$

Upshot: $\tilde{S}$, given as above, defines algebra morphisms $GL_q(2) \to GL_q(2)^{op}$, $SL_q(2) \to SL_q(2)^{op}$.

But $\det_q$ is a central invertible elt in $GL_q(2)$ and $SL_q(2)$. Therefore, $S\begin{pmatrix} a & b \\ c & d \end{pmatrix} := \det_q^{-1} \tilde{S}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and consequently $S(t) = t^{-1}$ give rise to alg. homom.

2. Second, we need to check antipode properties:
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix} = \begin{pmatrix} \det_q & 0 \\ 0 & \det_q \end{pmatrix} = \begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
$$\det_q \cdot \det_q^{-1} = 1 = \det_q^{-1} \cdot \det_q$$

■

Rmk: Note that $S^2\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & q^2 b \\ q^{-2}c & d \end{pmatrix}$

Lecture #6

• Coaction on quantum plane

We shall now finally establish a q-version of the comodule algebra structure on $\mathbb{K}[x,y]$ over $M(2)$.

Proposition 2: There exists a unique $M_q(2)$, $GL_q(2)$, $SL_q(2)$ comodule-algebra structure on the quantum plane $A=\mathbb{K}_q[x,y]$ s.t.
$$\Delta_A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} x \\ y \end{pmatrix}$$

▸ The fact that Δ_A gives rise to the algebra homomorphism is a direct computation, or equivalently follows from Prop 1 of Lecture 5.
The coassociativity & counity are checked the same way as for $q=1$
Finally, as $M_q(2) \longrightarrow SL_q(2)$ is an alg. homom, we also get result for $SL_q(2)$ ■

Rmk: Evoking the grading $\mathbb{K}_q[x,y] = \bigoplus_{n\geq 0} \mathbb{K}_q[x,y]_n$, we see that each graded component $\mathbb{K}_q[x,y]_n$ is a subcomodule of the quantum plane, and $\mathbb{K}_q[x,y]$ is the direct sum of these subcomodules of finite dimension.

Q-n Can we find a closed f-le for $\Delta_A(x^k y^\ell)$.
To answer the above question, we will take a quick detour to

• q-binomial coefficients (a.k.a. Gauss polynomials)

For $n \in \mathbb{N}$, set $\boxed{(n)_q := \frac{q^n-1}{q-1} = 1+q+q^2+ \_\_ + q^{n-1}}$

Define the q-factorial $\boxed{(n)!_q := (1)_q (2)_q \cdot \_\_ \cdot (n)_q = \frac{(q^n-1)(q^{n-1}-1) \cdot \_\_ \cdot (q-1)}{(q-1)^n}}$

Finally, for integers $0 \leq k \leq n$, define the q-binomial coeff:
$$\boxed{\binom{n}{k}_q := \frac{(n)!_q}{(k)!_q\,(n-k)!_q}}$$

Rmk: The above has a combinatorial interpretation for $q=p^r$, p-prime, $r \in \mathbb{Z}_{>0}$.
Namely, $\binom{n}{k}_q = \#\left\{\begin{matrix} k\text{-dimensional subspaces} \\ \text{of } n\text{-dim v. space over } \mathbb{F}_q \end{matrix}\right\} = \left| Gr_{\mathbb{F}_q}(k,n) \right|$

Lecture #6

Lemma 1: For $0 \le k \le n$, we have

a) $\binom{n}{k}_q = \binom{n}{n-k}_q$

b) $\binom{n}{k}_q = \binom{n-1}{k-1}_q + q^k \binom{n-1}{k}_q = \binom{n-1}{k}_q + q^{n-k}\binom{n-1}{k-1}_q \Leftarrow$ "q-Pascal identity"

c) $\binom{n}{k}_q$ is a polynomial in q with integral coeff-s, $\binom{n}{k}_{q=1} = \binom{n}{k}$

▶ a) Obvious.

b) Direct check.

c) Follows by induction from b). ∎

[Exercise: In view of the previous Rmk, give combinatorial proof of b).

The reason why it's relevant to us is:

Proposition 3: Let $yx = qxy$. Then $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k}_q x^k y^{n-k} \quad \forall n \ge 0$

▶ The proof is by induction on n. Base $n=0,1$ is obvious.
For the induction step, assume we know the result for $n<N$. Then:

$$(x+y)^N = (x+y)\cdot(x+y)^{N-1} = (x+y)\cdot\sum_{k=0}^{N-1}\binom{N-1}{k}_q x^k y^{N-k} = \sum_{k=0}^{N} \underbrace{\left(\binom{N-1}{k-1}_q + q^k\binom{N-1}{k}_q\right)}_{=\binom{N}{k}_q \text{ by Lemma 1b)}} x^k y^{N-k}$$

∎

We are now ready to answer Q-n from the previous page.

Lemma 2: $\Delta_A(x^k y^\ell) = \sum_{r=0}^{k}\sum_{s=0}^{\ell} q^{(k-r)s} a^r b^{k-r} c^s d^{\ell-s} \otimes x^{r+s} y^{k+\ell-r-s} \cdot \binom{k}{r}_{q^2}\binom{\ell}{s}_{q^2}$

▶ As $\Delta_A(x) = a\otimes x + b\otimes y$ and $yx = qxy$, $ba = qab$, we get by Prop 3:

$$\Delta_A(x^k) = (a\otimes x + b\otimes y)^k = \sum_{r=0}^{k}\binom{k}{r}_{q^2} a^r b^{k-r} \otimes x^r y^{k-r}$$

Similarly:

$$\Delta_A(y^\ell) = \sum_{s=0}^{\ell}\binom{\ell}{s}_{q^2} c^s d^{\ell-s} \otimes x^s y^{\ell-s}$$

Finally, we gain $q^{(k-r)s}$ as we move y^{k-r} to the right of x^s. ∎