Lecture #7

- Last time we ended up with q-binomial coefficients. An important notion for future is:

$$\boxed{\text{q-exponent} \qquad e_q(z) := \sum_{n\geq 0} \frac{z^n}{(n)!_q} \in \mathbb{k}[[z]]} \qquad (\leftarrow \text{assuming } q \neq \sqrt[?]{1})$$

Lemma 1: Let $yx = qxy$. Then $e_q(x+y) = e_q(x)\cdot e_q(y)$

▷ $$e_q(x)\cdot e_q(y) = \sum_{k\geq 0} \frac{x^k}{(k)!_q} \sum_{\ell\geq 0} \frac{y^\ell}{(\ell)!_q} = \sum_{n\geq 0} \sum_{k=0}^{n} \frac{1}{(k)!_q (n-k)!_q} x^k y^{n-k} = \underbrace{\sum_{n\geq 0} \frac{1}{(n)!_q} \underbrace{\sum_{k=0}^{n} \binom{n}{k}_q x^k y^{n-k}}_{=(x+y)^n \text{ by q-binomial formula}}}_{= e_q(x+y)}$$

∎

Note that when x,y commute, we do not have $e_q(x+y) = e_q(x)e_q(y)$. Thus, $e_q(x)^{-1} \neq e_q(-x)$

Lemma 2: $$e_q(z)^{-1} = \sum_{n\geq 0} (-1)^n q^{\frac{n(n-1)}{2}} \frac{z^n}{(n)_q!} \in \mathbb{k}[[z]]$$

The proof of this result is more important than the result itself. The idea is to apply Lemma 1 but to linear maps $\mathbb{k}[[z]] \to \mathbb{k}[[z]]$. To this end, consider

$$\boxed{\begin{aligned} Z &: \mathbb{k}[[z]] \longrightarrow \mathbb{k}[[z]], \quad f(z) \longmapsto z f(z) \\ \tau_q &: \mathbb{k}[[z]] \longrightarrow \mathbb{k}[[z]], \quad f(z) \longmapsto f(q\cdot z) \end{aligned}}$$

Then $\tau_q(Z f(z)) = qz\cdot f(qz) = q\cdot Z(\tau_q f(z))$, hence, (Z, τ_q) is $\mathrm{End}(\mathbb{k}[[z]])$-point of q-plane

Proof of Lemma 2

▷ Consider $(Z, -Z\tau_q)$ - another $\mathrm{End}(\mathbb{k}[[z]])$-point of the q-plane. Apply Lemma 1:

$$\boxed{e_q(Z(1-\tau_q)) = e_q(Z)\, e_q(-Z\tau_q)}$$

Applying this to $1 \in \mathbb{k}[[z]]$, we find:

$$\left.\begin{aligned} &LHS(1) = e_q(Z(1-\tau_q))1 = 1 \quad \text{since} \quad (1-\tau_q)1 = 0 \\ &RHS(1) = e_q(z)\cdot (e_q(-Z\tau_q)1) \overset{\text{use def-n}}{\underset{\text{of q-exp}}{=}} e_q(z)\cdot \left(\sum_{n\geq 0} (-1)^n q^{\frac{n(n-1)}{2}} \frac{z^n}{(n)!_q}\right) \end{aligned}\right\} \Rightarrow \text{desired f-le for } e_q(z)^{-1}$$

∎

Exercise: Use a similar argument applied to $\mathrm{End}(\mathbb{k}[[z]])$-point $(-Z\tau_q, a\tau_q)$ of the q-plane (here $a \in \mathbb{k}$) to prove the well-known f-le:

$$\boxed{(a-z)(a-qz)\dots(a-q^{n-1}z) = \sum_{k=0}^{n} (-1)^k \binom{n}{k}_q q^{\frac{k(k-1)}{2}} a^{n-k} z^k}$$

Lecture #7

• Lie algebras

Def: A *Lie algebra* is a vector space $\mathfrak{g}$ together with bilinear map "Lie bracket" $[\,,\,]: \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$ s.t.

1) $[x,y]=-[y,x]$ $\quad\forall x,y\in\mathfrak{g}$

2) $[[x,y],z]+[[y,z],x]+[[z,x],y]=0$ $\quad\forall x,y,z\in\mathfrak{g}$.

Some relevant definitions are:

- a subspace I of $\mathfrak{g}$ is an *ideal* if $[I,\mathfrak{g}]\subseteq I$
- a subspace $\mathfrak{a}$ of $\mathfrak{g}$ is a *Lie subalgebra* if $[\mathfrak{a},\mathfrak{a}]\subseteq\mathfrak{a}$
- a linear map $\mathfrak{g}_1 \xrightarrow{f} \mathfrak{g}_2$ b/w Lie alg-s is a *morphism* if $f([x,y])=[f(x),f(y)]$

Examples

1) Any associative algebra A becomes a Lie algebra through
$$[x,y]:=x\cdot y-y\cdot x \quad \forall x,y\in A.$$
(Kassel denotes it by $L(A)$)

2) Given two Lie algebras $\mathfrak{g}_1,\mathfrak{g}_2$, their *direct sum* $\mathfrak{g}_1\oplus\mathfrak{g}_2$ becomes a Lie algebra
$$[(x,y),(x',y')]:=([x,x'],[y,y']) \quad \forall x,x'\in\mathfrak{g}_1,\ y,y'\in\mathfrak{g}_2$$

3) Given a Lie algebra $\mathfrak{g}$, the *opposite* Lie algebra $\mathfrak{g}^{op}$ has the same v. space and
$$[x,y]^{op}=[y,x] \quad \forall x,y\in\mathfrak{g}.$$

4) Given a Lie algebra $\mathfrak{g}$ and its ideal I, the quotient vector space $\mathfrak{g}/I$ is equipped with a natural Lie alg. structure.

Of particular importance is the Lie algebras $\mathfrak{gl}(V)$ and $\mathfrak{sl}(V)$.

Def: (1) Given any vector space V, the Lie algebra *$\mathfrak{gl}(V)$* is the special case of 1) above with $A=\mathrm{End}(V)$.

(2) The Lie algebra *$\mathfrak{sl}(V)$* is a subalgebra of $\mathfrak{gl}(V)$ consisting of trace $=0$ elts

it's subalgebra as $tr(XY-YX)=0$ $\qquad$ assuming $\dim V<\infty$

Lecture #7

To every Lie algebra one can associate an algebra that governs its theory:

Def: The universal enveloping algebra of $\mathfrak{g}$, denoted by $\mathcal{U}(\mathfrak{g})$, is the algebra:

$$\mathcal{U}(\mathfrak{g}) = T(\mathfrak{g}) / \underset{\forall x,y\in\mathfrak{g}}{(xy - yx - [x,y])}$$

i.e. it's a quotient of the tensor algebra $T(\mathfrak{g}) = \bigoplus_{n\geq 0} \mathfrak{g}^{\otimes n}$ by the two-sided ideal generated by $xy - yx - [x,y]$ $\forall x,y\in\mathfrak{g}$

Composing $\mathfrak{g} \hookrightarrow T(\mathfrak{g}) \twoheadrightarrow \mathcal{U}(\mathfrak{g})$, we get an (injective) linear map

$$\boxed{\iota_{\mathfrak{g}} : \mathfrak{g} \longrightarrow \mathcal{U}(\mathfrak{g})}$$

The following is the key property of $\mathcal{U}(\mathfrak{g})$, which provides its universal property:

Proposition 1: For any associative algebra A and any Lie algebra morphism $f: \mathfrak{g} \to A$ there exists a unique algebra morphism $F: \mathcal{U}(\mathfrak{g}) \to A$ s.t.
$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\iota_{\mathfrak{g}}"] \arrow[rr, bend right, "f"'] & \mathcal{U}(\mathfrak{g}) \arrow[r, "F"] & A \end{tikzcd}$$
commutes

[**Exercise**: Prove this

The above implies that

$$\boxed{\mathrm{Hom}_{alg}(\mathcal{U}(\mathfrak{g}), A) = \mathrm{Hom}_{LieAlg}(\mathfrak{g}, A)}$$

Another important result on $\mathcal{U}(\mathfrak{g})$ is the PBW theorem.

Formulation 1 of PBW: Fix any basis $\{x_1, \dots, x_n\}$ of $\mathfrak{g}$. Then the ordered monomials $\{x_1^{k_1} \cdots x_n^{k_n} \mid k_1, \dots, k_n \in \mathbb{Z}_{\geq 0}\}$ form a basis of $\mathcal{U}(\mathfrak{g})$)

A bit more conceptual basis independent formulation of this result:

Formulation 2 of PBW: $\mathrm{gr}.\ \mathcal{U}(\mathfrak{g}) \simeq S(\mathfrak{g})$ as vector spaces

Here, $\mathcal{U}(\mathfrak{g})$ is a filtered algebra via $\deg(x) = 1$ $\forall x\in\mathfrak{g}$, and gr. refers to the associated graded of a filtered algebra. Finally, $S(\mathfrak{g})$ is the symmetric algebra in $\mathfrak{g}$ (i.e. polynomials in $\mathfrak{g}^*$).

Note: In particular, we see that $\iota_{\mathfrak{g}}$ is always injective.

Lecture #7

An important application of Proposition 1 is:

Corollary 1: a) For any Lie alg. morphism $f : \mathfrak{g}_1 \to \mathfrak{g}_2$, there is a unique algebra morphism $\mathcal{U}(f) : \mathcal{U}(\mathfrak{g}_1) \to \mathcal{U}(\mathfrak{g}_2)$ s.t.

$$\begin{tikzcd} \mathfrak{g}_1 \arrow[r, "f"] \arrow[d, "\imath_{\mathfrak{g}_1}"] & \mathfrak{g}_2 \arrow[d, "\imath_{\mathfrak{g}_2}"] \\ \mathcal{U}(\mathfrak{g}_1) \arrow[r, "\mathcal{U}(f)"] & \mathcal{U}(\mathfrak{g}_2) \end{tikzcd}$$

b) If $f_1 : \mathfrak{g}_1 \to \mathfrak{g}_2$, $f_2 : \mathfrak{g}_2 \to \mathfrak{g}_3$ are Lie alg. morphisms, then $\mathcal{U}(f_2 \circ f_1) = \mathcal{U}(f_2) \circ \mathcal{U}(f_1) : \mathcal{U}(\mathfrak{g}_1) \to \mathcal{U}(\mathfrak{g}_3)$.

c) For any Lie algebras $\mathfrak{g}_1, \mathfrak{g}_2$, we have $\mathcal{U}(\mathfrak{g}_1 \oplus \mathfrak{g}_2) \simeq \mathcal{U}(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2)$

This result allows for a formula-free conceptual proof of the key result.

Theorem 1: For any Lie algebra $\mathfrak{g}$ its universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ is naturally equipped with a unique Hopf alg. structure s.t.

$$\Delta(x) = x \otimes 1 + 1 \otimes x, \quad S(x) = -x, \quad \varepsilon(x) = 0 \quad \forall x \in \mathfrak{g}$$

As $\mathcal{U}(\mathfrak{g})$ is generated by $\mathfrak{g}$, the uniqueness is clear. However, the existence follows from the explicit description of Δ, S, ε on $\mathcal{U}(\mathfrak{g})$:

- $\Delta = \mathcal{U}(\mathrm{diag}) : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g} \oplus \mathfrak{g}) \simeq \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$

  where $\mathrm{diag} : \mathfrak{g} \to \mathfrak{g} \oplus \mathfrak{g}$, $x \mapsto (x, x)$

- $S = \mathcal{U}(\mathrm{op}) : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g}^{\mathrm{op}})$

  where $\mathrm{op} : \mathfrak{g} \to \mathfrak{g}^{\mathrm{op}}$, $x \mapsto -x$

- $\varepsilon = \mathcal{U}(0) : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\{0\}) \simeq \mathbb{k}$

  where $0 : \mathfrak{g} \to \{0\}$

Exercise: 1) Verify the above indeed makes $\mathcal{U}(\mathfrak{g})$ into a Hopf algebra.

2) Verify $\Delta(x_1 \cdot \ldots \cdot x_n) = \sum_{k=0}^{n} \sum_{\sigma - (k, n-k)\text{-shuffle}} x_{\sigma(1)} \ldots x_{\sigma(k)} \otimes x_{\sigma(k+1)} \ldots x_{\sigma(n)}$