Lecture #8

- Last time:

Lie algebra $\mathfrak{g}$ $\rightsquigarrow$ the universal enveloping algebra $U\mathfrak{g}$

becomes a Hopf algebra under

$$\boxed{\Delta(x) = x\otimes 1 + 1\otimes x, \quad S(x) = -x, \quad \varepsilon(x) = 0 \quad \forall x\in\mathfrak{g}}$$

Note that, as discussed in Week 1, given a bialgebra A and two A-modules $V_1 \otimes V_2$ one gets an A-module structure on $V_1\otimes V_2$. Likewise, given a Hopf alg. A and an A-module V, one gets an A-module structure on the dual space V^*. Combining this with the above f-las for Hopf alg. structure on $U\mathfrak{g}$, as well as observation that $\underbrace{\mathfrak{g}\text{-modules}}_{\text{v. space } V \text{ together with Lie alg. homom. } \mathfrak{g}\to \mathrm{End}(V)}$ are the same as $U\mathfrak{g}$-modules, we conclude

Corollary 1: a) Given $\mathfrak{g}$-modules V_1, V_2, their tensor product $V_1\otimes V_2$ is equipped with $\mathfrak{g}$-module structure via $x(v_1\otimes v_2) = x(v_1)\otimes v_2 + v_1\otimes x(v_2)$ $\quad v_1\in V_1, v_2\in V_2$

b) Given $\mathfrak{g}$-module V, its dual space V^* is equipped with $\mathfrak{g}$-module structure via $(x\varphi)(v) = -\varphi(xv)$ $\quad \varphi\in V^*, v\in V$

Let's now specialize from the general story to the simplest nontrivial case: $\mathfrak{sl}_2$.

- Lie algebra $\mathfrak{sl}_2$

- Basis: $e = \begin{pmatrix}0&1\\0&0\end{pmatrix}$, $h = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$, $f = \begin{pmatrix}0&0\\1&0\end{pmatrix}$ with the Lie bracket defined through

$$\boxed{[h,e] = 2e, \quad [h,f] = -2f, \quad [e,f] = h}$$

- The following rel-s can be verified by induction:

$$\boxed{\begin{aligned} &e^k h^l = (h-2k)^l e^k, \quad f^k h^l = (h+2k)^l f^k \\ &[e,f^k] = kf^{k-1}(h-k+1) = k(h+k-1)f^{k-1} \\ &[e^k, f] = ke^{k-1}(h+k-1) = k(h-k+1)e^{k-1} \end{aligned}}$$

- PBW thm implies that $\{e^i h^k f^l\}_{i,k,l\in\mathbb{Z}_{\geq 0}}$ - basis of $U(\mathfrak{sl}_2)$.

- The Casimir elt $C := ef + fe + \frac{1}{2}h^2 \in U(\mathfrak{sl}_2)$ is central

$$\left(\begin{aligned} &[C,e] = -eh - he + he + eh = 0 \\ &[C,h] = -2ef + 2ef + 2fe - 2fe + 0 = 0 \\ &[C,f] = hf + fh - hf - fh = 0 \end{aligned}\right)$$

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- If $\mathrm{char}(\Bbbk)=0$, then $\underbrace{Z(U(sl_2))}_{\text{center of } U(sl_2)} \simeq \Bbbk[C]$, i.e. center of $U(sl_2)$ is a polynomial algebra in C.

- Any finite-dimensional sl_2-module is completely reducible, i.e. isomorphic to $\oplus$ simple ones. The simple f.d. sl_2-modules are labelled by $n \in \mathbb{Z}_{\geq 0}$. For each $n \in \{0,1,2,\dots\}$, the corresponding module V_n is $(n+1)$-dimensional

Explicitly: $V_n = \Bbbk[x,y]_n$ = pol-s in x,y of homogeneous degree n
$\circlearrowleft$ sl_2 via $\boxed{e \mapsto x\frac{\partial}{\partial y},\ f \mapsto y\,\partial/\partial x,\ h \mapsto x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}}$

So: $\Bbbk[x,y] = \bigoplus_{n\geq 0} \Bbbk[x,y]_n$ encodes each f.d. sl_2-module once

- The Casimir C acts on V_n as a multiplication by $\frac{n(n+2)}{2}$

- Clebsch-Gordan F-la: $V_n \otimes V_m \simeq V_{n+m} \oplus V_{n+m-2} \oplus \dots \oplus V_{n-m}$ $\forall\, n\geq m\geq 0$
  - Implicit proof – through comparison of characters, i.e. $\mathrm{tr}(z^h)$.
  - Explicit proof – see Kassel's Ch V.5

Moreover, the above example fits into a context dual to comodule-algebras:

Def: Let H be a bialgebra, A – an algebra. A is called H-module-algebra if:
a) there is an action of H on A, i.e. $\mu_A: H\otimes A \to A$
b) the multiplication and unit $A\otimes A \to A$, $\Bbbk \to A$ are H-module morphisms.

Down-to-earth, the condition b) in Sweedler's notations means:

$$\boxed{x(ab) = \sum_{(x)} (x'a)(x''b) \quad , \quad x(1) = \varepsilon(x)\cdot 1}$$

Lemma 1: Given a Lie algebra $\mathfrak{g}$ and an algebra A, endowing A with an $U\mathfrak{g}$-module-algebra structure is equivalent to endowing A with a $\mathfrak{g}$-action via derivations of A (i.e. $x(ab) = x(a)\cdot b + a\cdot x(b)$)

Follows from Def and $\Delta(x) = x\otimes 1 + 1\otimes x$ $\forall\, x\in\mathfrak{g}$.

Therefore, evoking the above explicit realization of $V(n)$, we get:

Corollary 2: $\Bbbk[x,y]$ is an $U(sl_2)$-module-algebra

Rem: As we shall discuss later the Hopf alg-s $SL(2)$ and $U(sl_2)$ are dual.

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• **Quantum $U_q(\mathfrak{sl}_2)$**

We shall now introduce a q-deformation of $U(\mathfrak{sl}_2)$. To this end, fix $q \in \mathbb{k} \setminus \{\pm 1\}$. Set

$$\boxed{[n] = [n]_q := \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [n]! := [1] \cdot \ldots \cdot [n], \quad \begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]!\,[n-k]!}}$$

Rmk: This is more symmetric than $(n)_q$, $(n)_q!$, $\binom{n}{k}_q$ w.r.t. $q \leftrightarrow q^{-1}$, but is closely related, e.g. $[n] = q^{1-n}(n)_{q^2}$.

Def: $U_q(\mathfrak{sl}_2)$ is an associative algebra generated by $\{E, F, K^{\pm 1}\}$ subject to:

$$\boxed{K^{-1}K = 1 = KK^{-1}, \quad KE = q^2 EK, \quad KF = q^{-2} FK, \quad EF - FE = \frac{K - K^{-1}}{q - q^{-1}}}$$

$\boxed{Q}$: In which sense can we view $U_q(\mathfrak{sl}_2)$ as a deformation of $U(\mathfrak{sl}_2)$?

Note: While we could set $q = 1$ in $M_q(2)$ to get $M(2)$, we cannot follow that strategy now since the last rel-n has a pole at $q = 1$!

To get around the above issue, we shall now present a different realization of $U_q(\mathfrak{sl}_2)$ which will allow us to directly specialize $q = 1$:

Def: Define $\tilde{U}_q(\mathfrak{sl}_2)$ as an associative algebra generated by $\{E, F, K^{\pm 1}, L\}$ subject to the following relations:

$$\boxed{\begin{gathered} K^{-1}K = 1 = KK^{-1}, \quad KE = q^2 EK, \quad KF = q^{-2} FK, \quad EF - FE = L, \quad (q - q^{-1})L = K - K^{-1} \\ LE - EL = q(EK + K^{-1}E), \quad LF - FL = -q^{-1}(FK + K^{-1}F) \end{gathered}}$$

Lemma 2: The algebras $U_q(\mathfrak{sl}_2)$ and $\tilde{U}_q(\mathfrak{sl}_2)$ are isomorphic for any $q \neq \pm 1$

▷ There is clearly an algebra morphism $\varphi: U_q(\mathfrak{sl}_2) \to \tilde{U}_q(\mathfrak{sl}_2)$ s.t. $E \mapsto E, F \mapsto F, K^{\pm 1} \mapsto K^{\pm 1}$.
We claim that there is also an algebra morphism $\psi: \tilde{U}_q(\mathfrak{sl}_2) \to U_q(\mathfrak{sl}_2)$ s.t.

$$E \mapsto E, \quad F \mapsto F, \quad K^{\pm 1} \mapsto K^{\pm 1}, \quad L \mapsto [E, F].$$

To this end, we need to check compatibility with all rel-s. The first line is clear. As per the two in the 2nd line, let's check the 1st:

$$[\psi(L), \psi(E)] = [[E,F], E] = \frac{1}{q - q^{-1}} [K - K^{-1}, E] = \frac{(q^2 - 1)EK + (q^2 - 1)K^{-1}E}{q - q^{-1}} = q\big(\psi(E)\cdot\psi(K) + \psi(K^{-1})\cdot\psi(E)\big)$$

Clearly φ and ψ are opposite to each other. ∎

The benefit of using $\tilde{U}_q(\mathfrak{sl}_2)$ is that one can directly set $q \mapsto 1$ in all the defining rel-s there. The result is closely related to $U(\mathfrak{sl}_2)$ ↘

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Lemma 3: For $q=1$, we have
$$\widetilde{U}_1(\mathfrak{sl}_2) \simeq U(\mathfrak{sl}_2)[K]/(K^2-1) \implies U(\mathfrak{sl}_2) \simeq \widetilde{U}_1(\mathfrak{sl}_2)/(K-1)$$

▸ Specializing $q \mapsto 1$, we see that $\widetilde{U}_1(\mathfrak{sl}_2)$ is an algebra generated by $\{E, F, K^{\pm 1}, L\}$ s.t.

K-central, $[E,F]=L$, $K-K^{-1}=0$

$$[L,E] = \underbrace{EK + K^{-1}E}_{=2EK}, \quad [L,F] = \underbrace{-(FK + K^{-1}F)}_{=-2FK}$$

↑ b/c $K^{-1}=K$ and K-central

Hence, the assignment $E \mapsto e\cdot K$, $F \mapsto f$, $K \mapsto K$, $L \mapsto h\cdot K$ give rise to
$$\widetilde{U}_1(\mathfrak{sl}_2) \xrightarrow{\sim} U(\mathfrak{sl}_2)[K]/(K^2-1) \qquad \leftarrow \text{check it!}$$

The second isomorphism $U(\mathfrak{sl}_2) \simeq \widetilde{U}_1(\mathfrak{sl}_2)/(K-1)$ is now obvious ∎

Rmk: Another approach to a similar quantum-to-classical limits is based on integral forms. To this end, let us view q as a formal variable not an elt of $\Bbbk$, i.e. $U_q(\mathfrak{sl}_2)$ is now defined over $\Bbbk(q)$. Then, one may wish to find an appropriate $\Bbbk[q,q^{-1}]$-subalgebra $\mathfrak{U}_q(\mathfrak{sl}_2)$ s.t. $\mathfrak{U}_q(\mathfrak{sl}_2) \otimes_{\Bbbk[q,q^{-1}]} \Bbbk(q) \simeq U_q(\mathfrak{sl}_2)$ and then specialize $q \mapsto 1$ in $\mathfrak{U}_q(\mathfrak{sl}_2)$ in the sense $\mathfrak{U}_q(\mathfrak{sl}_2) \otimes_{\Bbbk[q,q^{-1}]} \Bbbk$ where $\Bbbk[q,q^{-1}] \to \Bbbk$ sends $q^{\pm 1} \mapsto 1$.