Lecture #9

- Overview Lemmas 2-3 from Lecture 8 that we rushed through

<u>Plan for today</u>: 1) PBW theorem for $U_q(\mathfrak{sl}_2)$
2) $U_q(\mathfrak{sl}_2)$ - domain.

We start from the following straightforward result:

Lemma 1: For any $n\in\mathbb{Z}$, $m\in\mathbb{Z}_{>0}$:

a) $K^nE^m = q^{2nm}\cdot E^mK^n$, $K^nF^m = q^{-2nm}F^mK^n$

b) $[E,F^m] = [m]\cdot F^{m-1}\cdot[K;1-m] = [m]\cdot[K;m-1]\cdot F^{m-1}$

c) $[F,E^m] = -[m]\cdot E^{m-1}\cdot[K;m-1] = -[m]\cdot[K;1-m]\cdot E^{m-1}$

where we set
$$[K;a] := \frac{Kq^a - K^{-1}q^{-a}}{q-q^{-1}} \qquad \forall a\in\mathbb{Z}$$

► a) Obvious.

b) Proof is by induction. Base case $m=1$: from the defining relation $[E,F]=[K;0]$.
As per the induction step:
$$EF^m = EF^{m-1}\cdot F \xlongequal{\text{ind. hypothesis}} (F^{m-1}E + [m-1]F^{m-2}[K;2-m])F =$$
$$= F^mE + F^{m-1}\cdot\frac{K-K^{-1}}{q-q^{-1}} + [m-1]F^{m-1}\cdot\frac{Kq^{-m}-K^{-1}q^m}{q-q^{-1}} =$$
$$= F^mE + F^{m-1}\cdot\frac{(q-q^{-1})(K-K^{-1}) + (q^{m-1}-q^{1-m})(Kq^{-m}-K^{-1}q^m)}{(q-q^{-1})^2}$$
$$= F^mE + F^{m-1}\cdot\frac{K(q-q^{1-2m}) - K^{-1}(q^{2m-1}-q^{-1})}{(q-q^{-1})^2} = F^mE + F^{m-1}\cdot[m]\cdot[K;1-m].$$

This proves the first equality in b). To get the 2nd one, use a).

c) Analogous. ■

Theorem 1 (PBW thm): The set $\{F^aK^nE^b \mid a,b\in\mathbb{Z}_{\geq 0}, n\in\mathbb{Z}\}$ is a $\Bbbk$-basis of $U_q(\mathfrak{sl}_2)$

As always with PBW thms, spanning property is easy, while lin. independence is harder. For the latter, we will use one of the classical arguments involving faithful representations (the other ones involve e.g. Diamond Lemma or Ore Extensions)

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Proof of Theorem 1

- **Spanning property**

Let $V := \mathrm{Span}_{\Bbbk} \langle F^a K^n E^b \rangle$

$$\boxed{\text{Claim 1: } V \text{ is stable under left multiplication by } U_q(\mathfrak{sl}_2)}$$

As $U_q(\mathfrak{sl}_2)$ is generated by $E, F, K^{\pm 1}$, it suffices to verify V is stable under those

(1) $F \cdot F^a K^n E^b = F^{a+1} K^n E^b \in V$

(2) $K^{\pm 1} \cdot F^a K^n E^b = q^{\mp 2a} F^a \cdot K^{n\pm 1} \cdot E^b \in V$

(3) $E \cdot F^a K^n E^b \overset{\text{Lemma 1(b)}}{=\!=\!=} (F^a E + F^{a-1} \cdot [a] \cdot [K; 1-a]) \cdot K^n E^b =$

$$= q^{-2n} \cdot F^a K^n E^{b+1} + [a] \cdot F^{a-1} \cdot \left(\frac{Kq^{1-a} - K^{-1} q^{a-1}}{q - q^{-1}} K^n \right) \cdot E^b \in V$$

But $V \ni 1 = F^0 K^0 E^0 \underset{\text{Claim 1}}{\Longrightarrow} U_q(\mathfrak{sl}_2) \cdot 1 = U_q(\mathfrak{sl}_2) \subseteq V \Rightarrow V = U_q(\mathfrak{sl}_2)$

This proves that those ordered monomials $F^a K^n E^b$ span $U_q(\mathfrak{sl}_2)$.

- **Linear independence**

To prove lin. indep. of $F^a K^n E^b$ we shall construct an action of $U_q(\mathfrak{sl}_2)$ on $\Bbbk[x, y, z^{\pm 1}]$

Claim 2: The following operations on $A := \Bbbk[x, y, z^{\pm 1}]$ satisfy defining rel-s of $U_q(\mathfrak{sl}_2)$

$$\left.\begin{aligned}
\tilde{F} &: y^a z^n x^b \longmapsto y^{a+1} z^n x^b \\
\tilde{K}^{\pm 1} &: y^a z^n x^b \longmapsto q^{\mp 2a} \cdot y^a z^{n\pm 1} x^b \\
\tilde{E} &: y^a z^n x^b \longmapsto q^{-2n} \cdot y^a z^n x^{b+1} + [a] \cdot y^{a-1} \cdot \frac{z q^{1-a} - z^{-1} q^{-1+a}}{q - q^{-1}} \cdot z^n \cdot x^b
\end{aligned}\right\} \text{compare them to (1)–(3) above!}$$

Before giving a proof of this Claim, let us first explain how it implies the lin. independence of $F^a K^n E^b$. To this end, we note:

$$\tilde{E}^b(1) = x^b, \quad \tilde{K}^n(x^b) = z^n x^b, \quad \tilde{F}^a(z^n x^b) = y^a z^n x^b$$

$$\Downarrow$$

$$\boxed{\underbrace{F^a K^n E^b (1)}_{\text{action of } F^a K^n E^b \text{ on } 1 \in A} = \tilde{F}^a \tilde{K}^n \tilde{E}^b (1) = y^a z^n x^b}$$

But $\{y^a z^n x^b \mid a, b \in \mathbb{Z}_{\geq 0}, n \in \mathbb{Z}\}$ – lin. indep. elements of A (actually a basis) hence, $\{F^a K^n E^b\}$ are lin. indep. elements of $U_q(\mathfrak{sl}_2)$. This completes the proof of Theorem 1, modulo a direct check of Claim 2

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Proof of Claim 2

1) $\tilde{K}\tilde{K}^{-1} = \mathrm{Id}_A = \tilde{K}^{-1}\tilde{K}$ is clear.

2) $\tilde{K}\tilde{F}: y^a z^u x^b \mapsto q^{-2(a+1)} y^{a+1} z^{u+1} x^b$
$\tilde{F}\tilde{K}: y^a z^u x^b \mapsto q^{-2a} y^{a+1} z^{u+1} x^b$ $\Big\}\Rightarrow \tilde{K}\tilde{F} = q^{-2}\tilde{F}\tilde{K}$

3) $\tilde{K}\tilde{E}: y^a z^u x^b \mapsto q^{-2u-2a} y^a z^{u+1} x^{b+1} + [a]\cdot q^{-2a+2}\cdot y^{a-1}\cdot \dfrac{z^2 q^{1-a} - q^{a-1}}{q-q^{-1}} z^u x^b$
$\tilde{E}\tilde{K}: y^a z^u x^b \mapsto q^{-2u-2a-2} y^a z^{u+1} x^{b+1} + [a]\cdot q^{-2a}\cdot y^{a-1}\cdot \dfrac{z q^{1-a} - z^{-1} q^{a-1}}{q-q^{-1}} z^{u+1} x^b$ $\Big\}\Rightarrow \tilde{K}\tilde{E} = q^2\tilde{E}\tilde{K}$

4) $\tilde{E}\tilde{F}: y^a z^u x^b \mapsto q^{-2u}\cdot y^{a+1} z^u x^{b+1} + [a+1]\cdot y^a \cdot \dfrac{z q^{-a} - z q^{a}}{q-q^{-1}} z^u x^b$
$\tilde{F}\tilde{E}: y^a z^u x^b \mapsto q^{-2u}\cdot y^{a+1} z^u x^{b+1} + [a]\cdot y^a \cdot \dfrac{z q^{1-a} - z^{-1} q^{a}}{q-q^{-1}} z^u x^b$ $\Big\}\Rightarrow$

$$\Rightarrow [\tilde{E},\tilde{F}]: y^a z^u x^b \mapsto y^a \underbrace{\left([a+1]\cdot\frac{z q^{-a} - z q^{a}}{q-q^{-1}} - [a]\cdot\frac{z q^{1-a} - z^{-1} q^{a-1}}{q-q^{-1}}\right)}_{=\frac{z q^{-2a} - z^{-1} q^{2a}}{q-q^{-1}}}\cdot z^u x^b$$

$$\Downarrow$$

$$[\tilde{E},\tilde{F}] = \frac{\tilde{K}-\tilde{K}^{-1}}{q-q^{-1}}$$

■

Upshot: The key part of the above proof of PBW thm relied on the faithful representation of $U_q(\mathfrak{sl}_2)$, namely, on commutative pol-s in $x, y, z^{\pm 1}$

Corollary 1: Let U_q^-, U_q^0, U_q^+ be the subalgebras of $U_q(\mathfrak{sl}_2)$ generated by $\{F\}$, $\{K^{\pm 1}\}$, $\{E\}$, respectively. Then:

a) $U_q^+ \simeq \Bbbk[E]$, $U_q^- \simeq \Bbbk[F]$, $U_q^0 \simeq \Bbbk[K^{\pm 1}]$

b) The multiplication map gives rise to isomorphism $U_q^- \otimes U_q^0 \otimes U_q^+ \xrightarrow{\sim} U_q(\mathfrak{sl}_2)$

[Exercise: Prove the corollary above.

Our second main result for today is:

Theorem 2: The algebra $U_q(\mathfrak{sl}_2)$ is a domain (i.e. $u\cdot v = 0 \Rightarrow u = 0$ or $v = 0$)

The proof will rely on the following mandatory (unpleasant) exercise.

[Exercise: $E^m F^n = \sum_{i=0}^{\min(m,n)} \begin{bmatrix} m \\ i \end{bmatrix}\cdot\begin{bmatrix} n \\ i \end{bmatrix}\cdot [i]!\cdot F^{n-i}\cdot \prod_{j=1}^{i} [K; i+j-(m+n)]\cdot E^{m-i}$

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- Evoking $\begin{cases} KE = q^2 EK = E\cdot(q^2K) \\ KF = q^{-2}FK = F\cdot(q^{-2}K)\end{cases}$, one gets $\begin{cases} p(K)\cdot E = E\cdot p(q^{+2}K) \\ p(K)\cdot F = F\cdot p(q^{-2}K)\end{cases}$ $\forall\, p(x)\in \Bbbk[x,x^{-1}]$

This motivates to define algebra automorphisms

$$\boxed{\gamma_i : U_q^0 \xrightarrow{\sim} U_q^0 \quad \text{s.t. } K\mapsto q^iK \qquad \text{for any } i\in\mathbb{Z}}$$

- According to Thm 1, any elt of $U_q(\mathfrak{sl}_2)$ can be uniquely written as a sum of terms $\{F^a\cdot h\cdot E^b \mid a,b\in\mathbb{Z}_{\geq 0},\ h\in U_q^0\}$. We shall order such terms via:

$$\boxed{F^a h E^b > F^{a'}h'E^{b'} \text{ if } a>a' \text{ or } a=a',\ b>b'}$$

This allows to speak about the leading term of any $u\in U_q(\mathfrak{sl}_2)$.

► Proof of Theorem 2.

Assume that $u,v\in U_q(\mathfrak{sl}_2)$ are nonzero elements with $u\cdot v=0$.

Let $F^a\cdot h\cdot E^b$ and $F^{a'}h'E^{b'}$ be the leading terms of u and v, respectively $\begin{pmatrix} h\neq 0\\ h'\neq 0\end{pmatrix}$

$$\boxed{\text{Claim 3: The leading term of } u\cdot v \text{ is } F^{a+a'}\cdot\gamma_{-2a'}(h)\gamma_{-2b}(h')\cdot E^{b+b'}.}$$

As γ_i are automorphisms and $U_q^0\simeq \Bbbk[K^{\pm 1}]$ is a domain, this proves Thm 2.
It remains to prove the Claim above:

▷ Proof of Claim 3

Using the Exercise from the bottom of p.3, we get

$$F^ahE^b\cdot F^{a'}h'E^{b'} = \sum_{i=0}^{\min(b,a')} F^a\cdot h\cdot F^{a'-i}\cdot \underbrace{h_i}_{\text{shorthand for what appears in Exercise}}\cdot E^{b-i}h'E^{b'} =$$

$$= \sum_{i=0}^{\min(b,a')} F^{a+a'-i}\cdot\gamma_{-2(a'-i)}(h)\cdot h_i\cdot \gamma_{-2(b-i)}(h')\cdot E^{b+b'-i}$$

Clearly, the leading term corresponds to $i=0$, in which case $h_0=1$ and we get $F^{a+a'}\cdot\gamma_{-2a'}(h)\gamma_{-2b}(h')E^{b+b'}$. Likewise, taking products of other terms of u & v will contain terms which are all smaller than the one above $\square$

$\blacksquare$