Lecture #10

Goal for today (and next time): Classify all finite dimensional modules of $U_q(\mathfrak{sl}_2)$ for $q \neq \sqrt[?]{1}$.

Def: a) Given a $U_q(\mathfrak{sl}_2)$-module V, an element $v \in V \setminus \{0\}$ is called a *highest weight vector of weight λ* if $E(v)=0$ and $K(v)=\lambda \cdot v$

b) V is called a *highest weight module* of highest weight λ if it is generated by a highest weight vector v of weight λ.

Lemma 1: Let V be a fin. dim. $U_q(\mathfrak{sl}_2)$-module. Then E, F act nilpotently on V.

- Let's first present a short proof for algebraically closed $\mathbb{K}$.

If E is not nilpotent, then $\exists \lambda \neq 0$, $v \in V \setminus \{0\}$, s.t. $E(v)=\lambda \cdot v$. But then $E(K^n v) = q^{-2n} K^n (Ev) = q^{-2n} \lambda \cdot K^n v \Rightarrow K^n v$ is an eigenvector of eigenvalue $q^{-2n}\lambda$. (it's not zero as $(K^{-1})^n K^n v = v$)

As $q \neq \sqrt[?]{1}$, all these numbers $\{\lambda, q^{-2}\lambda, q^{-4}\lambda, \dots\}$ are pairwise distinct.

Basic Result (exercise): eigenvectors with distinct eigenvalues are linearly indep.

But this would contradict $\dim V < \infty$. Hence, E is nilpotent. Same for F.

- Lets now generalize this argument to an arbitrary field $\mathbb{K}$

We start from the basic algebraic result:

$$V = \bigoplus_{\substack{f\text{-irred} \\ \text{pol. in } \mathbb{K}[x]}} V_{(f)}, \quad V_{(f)} := \{v \in V \mid f(E)^n v = 0 \text{ for } n \gg 0\}$$

As $KE = q^2 EK$, we get $E^r V_{(f)} \subseteq V_{(f_{-2r})}$ $\forall r \geq 0$, where $g_{2a}(x) := g(q^{2a} \cdot x)$ (f_{-2r} clearly is irreducible)

As V is fin. dim., the above sum is finite. Hence, by above, $V(f_{-2r}) = V(f_{-2s})$.

Therefore, $f_{-2r} = f_{-2s} \cdot \underset{\in \mathbb{K}^*}{c}$. As the constant term of f is not zero (unless $f(x)=x$) we get that constant terms of f_{-2r} & f_{-2s} coincide $\Rightarrow c=1$. But then comparing the leading terms, we find $q^{-2rn} = q^{-2sn}$ with $n = \deg(f)$

Since $q \neq \sqrt[?]{1}$, the latter is impossible. Therefore, $V_{(f)}=0$ unless $f(x)=x$, so that E-nilpotent. Same applies to F

Lemma 2: Assume $\mathrm{char}(\mathbb{K})\neq 2$, $q\neq \sqrt[?]{1}$. Let V be a fin. dim. $U_q(\mathfrak{sl}_2)$-module. Then $V=\bigoplus_{\lambda\in\mathbb{K}^\times} V_\lambda$, where $V_\lambda=\{v\in V \mid Kv=\lambda\cdot v\}$. If $V_\lambda\neq 0 \Rightarrow \lambda\in \pm q^{\mathbb{Z}}$

▹ Note that the above decomposition of V is equivalent to the action of K being diagonalizable. The latter is equivalent to the minimal polynomial of K-action to split into linear factors with multiplicity 1. For the latter, it suffices to find some nonzero pol-l that splits as a product of linear factors with multiplicity 1 and vanishes on the K-action.

By Lemma 1, $\exists N$ s.t. $F^N V=0$. Consider the following $h_r\in U_q\ \forall r\geq 0$

$$h_r:=\prod_{j=-(r-1)}^{(r-1)}[K;r-N+j] \quad \text{with } h_0:=1$$

Exercise: $F^{N-r}h_r V=0 \quad \forall\, 0\leq r\leq N$.

In particular, $h_N V=0$. But $h_N=\prod_{j=-(N-1)}^{N-1}\frac{Kq^j-K^{-1}q^{-j}}{q-q^{-1}}$. Therefore, we get:

$$\Big[\prod_{j=-(N-1)}^{N-1}(K-q^{-j})(K+q^{-j})\Big]V=0$$

It remains to note $\pm q^j\neq \pm q^{j'}$ unless $j=j'$ and signs coincide. ∎

Rmk: The assumption $\mathrm{char}(\mathbb{K})\neq 2$ is essential as we have the following counterexample to Lemma 2 otherwise: $V=\mathbb{K}^2$, $E\mapsto 0$, $F\mapsto 0$, $K^{\pm 1}\mapsto\begin{pmatrix}1&1\\0&1\end{pmatrix}$

Main Question we shall address next time: Provide a full classification of fin. dim. $U_q(\mathfrak{sl}_2)$-modules when $\mathrm{char}\,\mathbb{K}\neq 2$, $q\neq\sqrt[?]{1}$.

The treatment shall crucially rely on ∞-dim Verma modules.

Def: For $\lambda\neq 0$, the Verma module over $U_q(\mathfrak{sl}_2)$ is $M(\lambda):=U_q(\mathfrak{sl}_2)/(E, K-\lambda)$ (left ideal)

Lemma 3 (universal property): For any $U_q(\mathfrak{sl}_2)$-module V there exist a bijection $\mathrm{Hom}_{U_q(\mathfrak{sl}_2)}(M(\lambda),V)\xrightarrow{1\text{-to-}1}\{\text{highest weight vectors of } V \text{ of weight } \lambda\}$

$$\varphi \longmapsto \varphi(1)$$

Exercise: Prove this Lemma.

Lecture #10

Applying the triangular decomposition

$$U_q^- \otimes U_q^0 \otimes U_q^+ \xrightarrow{\sim} U_q(sl_2)$$

we see that $\{F^i \mid i \geq 0\}$ form a basis of $M(\lambda)$. Combining this with Lemma 1 of Lecture 9, we obtain explicit f-las for $M(\lambda)$.

Corollary: $M(\lambda)$ has the basis $v_0, v_1, v_2, \dots$ s.t.

$$K(v_i) = \lambda q^{-2i} v_i$$

$$F(v_i) = v_{i+1}$$

$$E(v_i) = \begin{cases} 0, & i = 0 \\ [i] \cdot \dfrac{\lambda q^{1-i} - \lambda^{-1} q^{-1+i}}{q - q^{-1}} \cdot v_{i-1}, & i > 0 \end{cases}$$