Lecture #11

Today we shall classify finite-dimensional $U_q(\mathfrak{sl}_2)$-modules, $q \neq \sqrt[?]{1}$, using Verma modules

Lemma 1: Let $\lambda \in \mathbb{k}^*$, $q \neq \sqrt[?]{1}$. Then:

a) If $\lambda \notin \pm q^{\mathbb{Z}_{\geq 0}}$, then $M(\lambda)$ is simple.

b) If $\lambda = \pm q^n$ with $n \in \mathbb{Z}_{\geq 0}$, then $\{v_i \mid i \geq n+1\}$ span a $U_q(\mathfrak{sl}_2)$-submodule of $M(\lambda)$ isomorphic to $M(q^{-2n-2}\lambda)$. The latter is the only nontrivial submodule of $M(\lambda)$.

▷ Recall that K acts on the basis $\{v_i \mid i \geq 0\}$ of $M(\lambda)$ via $K v_i = \lambda q^{-2i} \cdot v_i$. As $q \neq \sqrt[?]{1}$, all these eigenvalues are pairwise distinct. Hence, we can use

Linear Algebra Lemma: Given a vector space V and a diagonalizable linear operator $\alpha: V \to V$ with an eigenbasis $\{v_a\}$ of V s.t. $\alpha(v_a) = \lambda_a \cdot v_a$, $\lambda_a = \lambda_b \Leftrightarrow a = b$, any vector subspace $U \subseteq V$ satisfying $\alpha(U) \subseteq U$ is a span of some set of v_a's

Exercise

Thus, any submodule $0 \neq M' \neq M(\lambda)$ contains at least one of v_i. Next, as $F v_k = v_{k+1}$ $\forall k$, we conclude $M' \ni v_i, v_{i+1}, \ldots$ On the other hand, recall that $E v_k \overset{k>0}{=} [k] \cdot \frac{\lambda q^{1-i} - \lambda^{-1} q^{-1+i}}{q - q^{-1}} \cdot v_{k-1}$. Thus, if i is the minimal elt s.t. $v_i \in M'$, then either $i = 0$ (but then $M' = M(\lambda)$, contradiction) or $\lambda q^{1-i} = \lambda^{-1} q^{-1+i}$ (using $[i] \neq 0$). The latter is equivalent to $\lambda^2 = q^{2(i-1)} \Leftrightarrow \lambda = \pm q^{i-1} \overset{n=i-1}{=\!=} \pm q^n$.

This immediately implies part a). It also implies part b) except for the statement that $\mathrm{Span}\{v_{n+1}, v_{n+2}, \ldots\} \simeq M(q^{-2n-2}\lambda)$. However, v_{n+1} is a highest weight vector of weight $q^{-2n-2}\lambda$, it generates all the above span as $F v_k = v_{k+1}$, hence $M(q^{-2n-2}\lambda) \twoheadrightarrow \mathrm{Span}\{v_{n+1}, v_{n+2}, \ldots\}$. The latter is now clearly an isomorph, just by looking at bases of both $U_q(\mathfrak{sl}_2)$-modules ◼

Corollary 1: For each $\varepsilon \in \{\pm\}$, $n \in \mathbb{Z}_{\geq 0}$, the quotient $M(\varepsilon q^n)/M(\varepsilon q^{-n-2})$ is an irreducible $U_q(\mathfrak{sl}_2)$-module of dimension $n+1$

We shall use $L(n, \pm)$ to denote the above quotient $M(\pm q^n)/M(\pm q^{-n-2})$. It has a basis $v_0, v_1, \ldots, v_n$ (images of such vectors in $M(\pm q^n)$).

Proposition 1: Assume $q \neq \sqrt[?]{1}$, char $\mathbb{k} \neq 2$.

a) For any $n \in \mathbb{Z}_{\geq 0}$, there are two $(n+1)$-dimensional $U_q(\mathfrak{sl}_2)$-modules $L(n,+)$ with a basis $\{v_0, v_1, \dots, v_n\}$, $L(n,-)$ with a basis $\{v_0', v_1', \dots, v_n'\}$ s.t.

$$\boxed{\begin{array}{l} K(v_i) = q^{n-2i} v_i, \quad F(v_i) = \begin{cases} v_{i+1}, & i<n \\ 0, & i=n \end{cases}, \quad E(v_i) = \begin{cases} 0, & i=0 \\ [i]\cdot[n+1-i]v_{i-1}, & i>0. \end{cases} \\ K(v_i') = -q^{n-2i} v_i', \quad F(v_i') = \begin{cases} v_{i+1}', & i<n \\ 0, & i=n \end{cases}, \quad E(v_i') = \begin{cases} 0, & i=0 \\ -[i]\cdot[n+1-i]\, v_{i-1}', & i>0 \end{cases} \end{array}}$$

Both $L(n,\pm)$ are simple modules.

b) Any simple $(n+1)$-dimensional $U_q(\mathfrak{sl}_2)$-module is isomorphic to either of $L(n,\pm)$.

▸ a) The explicit f-las follow by specializing $\lambda \mapsto \pm q^n$ in the f-las for the action of $U_q(\mathfrak{sl}_2)$ on $M(\lambda)$.

b) By Lecture #10, we know that $V = \bigoplus_{\lambda \in \mathbb{k}^*} V_\lambda$ with $V_\lambda = \{v \in V \mid K(v) = \lambda \cdot v\}$.

As $E(V_\lambda) \subseteq V_{q^2\lambda}$ $\forall \lambda$, $q \neq \sqrt[?]{1}$, and $\dim V < \infty$, we see that $\exists \lambda$ s.t. $\begin{cases} V_\lambda \neq 0 \\ V_{q^2\lambda} = 0 \end{cases}$.

But then picking any $v \in V_\lambda \setminus \{0\}$, we get a $U_q(\mathfrak{sl}_2)$-module morphism

$$\pi_v : \begin{array}{ccc} M(\lambda) & \longrightarrow & V \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ v_0 & \longmapsto & v \end{array}$$

It's nonzero, V - simple, hence π_v - surjective. But then being a quotient of Verma $M(\lambda)$, we can apply Lemma 1 ∎

The key result for today is:

Theorem 1: Assume $q \neq \sqrt[?]{1}$, char $\mathbb{k} \neq 2$. Then any finite-dimensional $U_q(\mathfrak{sl}_2)$-module is semisimple, i.e. $\simeq \bigoplus$ simple ones.

Thus combining Thm 1 & Prop 1, we obtain a full classification of f.d. $U_q(\mathfrak{sl}_2)$-modules. Note that it looks very alike that for a Lie algebra $\mathfrak{sl}_2$ over $\mathbb{C}$, besides for extra $\pm$ feature.

One of the classical proofs of the theorem above (again similar to $\mathfrak{sl}_2$) uses the central element "quantum Casimir" that we discuss next.

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Def: The quantum Casimir of $U_q(\mathfrak{sl}_2)$ is defined as:

$$C = FE + \frac{Kq + K^{-1}q^{-1}}{(q-q^{-1})^2} = EF + \frac{Kq^{-1} + K^{-1}q}{(q-q^{-1})^2}$$

follows from $EF - FE = \frac{K-K^{-1}}{q-q^{-1}}$

Lemma 2: a) C is central

b) For any $\lambda \in \Bbbk^*$, C acts on $M(\lambda)$ as multiplication by $\frac{\lambda q + \lambda^{-1}q^{-1}}{(q-q^{-1})^2}$

▶ a) $EC = EFE + E \cdot \frac{Kq + K^{-1}q^{-1}}{(q-q^{-1})^2} = EFE + \frac{Kq^{-1} + K^{-1}q}{(q-q^{-1})^2} E = CE$

$FC = FEF + F\frac{Kq^{-1} + K^{-1}q}{(q-q^{-1})^2} = FEF + \frac{Kq + K^{-1}q^{-1}}{(q-q^{-1})^2} F = CF$ $\Rightarrow$ C-central

$KC = CK$ obviously

b) $C(v_0) = \underbrace{FE(v_0)}_{=0} + \frac{Kq + K^{-1}q^{-1}}{(q-q^{-1})^2}(v_0) = \frac{\lambda q + \lambda^{-1}q^{-1}}{(q-q^{-1})^2} v_0$

As $v_k = F^k v_0$ and $CF = FC$ by a), we get $Cv_k = CF^k v_0 = F^k C v_0 = \frac{\lambda q + \lambda^{-1}q^{-1}}{(q-q^{-1})^2} v_k$ ∎

Lemma 3: a) C acts on $M(\lambda), M(\mu)$ by the same scalars iff $\lambda = \mu$ or $\lambda = q^{-2}\mu^{-1}$.

b) C acts on two simple fin. dim. modules L_1, L_2 by the same scalars iff $L_1 \simeq L_2$.

▶ a) Using Lemma 2b), we get $\lambda q + \lambda^{-1}q^{-1} = \mu q + \mu^{-1}q^{-1} \Leftrightarrow (\lambda - \mu)q = \frac{(\lambda-\mu)q^{-1}}{\lambda\mu}$.

Hence, either $\lambda = \mu$ or $\lambda\mu = q^{-2}$.

b) As L_i is a quotient of $M(\varepsilon_i q^{n_i})$ for $i \in \{1,2\}$, where $\varepsilon_i \in \{\pm 1\}$, $n_i \in \mathbb{Z}_{\geq 0}$, we see that C acts on L_i by the same constant as on $M(\varepsilon_i q^{n_i})$.

Now applying part a) to $M(\varepsilon_1 q^{n_1}), M(\varepsilon_2 q^{n_2})$, we see that either $\varepsilon_1 q^{n_1} = \varepsilon_2 q^{n_2}$ ($\Rightarrow L_1 \simeq L_2$) or $\varepsilon_1 q^{n_1} = \varepsilon_2^{-1} q^{-2-n_2}$. But the latter is impossible as $q \neq \sqrt[?]{1}$ and both $n_1, n_2 \geq 0$.

∎

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Proof of Theorem 1

Let V be a finite dimensional $U_q(\mathfrak{sl}_2)$-module. Choose any Jordan-Hölder filtration $0=V_0\subset V_1\subset V_2\subset\ldots\subset V_{m-1}\subset V_m=V$ by $U_q(\mathfrak{sl}_2)$-submodules, so that each V_i/V_{i-1} is irreducible, i.e. $\simeq L(n_i,\pm)$ for some $n_i\in\mathbb{Z}_{\geq 0}$. Then, the quantum Casimir acts by a certain constant ζ_i on each V_i/V_{i-1}. Thus, $(C-\zeta_i)V_i\subseteq V_{i-1}$ and hence $\left(\prod_{i=1}^m C-\zeta_i\right)V=0$. This implies the decomposition

$$\boxed{V=\bigoplus_{\zeta} V_{(\zeta)}\ ,\ \text{where } V_{(\zeta)}=\{v\in V\mid (C-\zeta)^N v=0 \text{ for } N\gg 1\}} \leftarrow \text{generalized eigenspace decomposition}$$

Since C is central, we see that each $V_{(\zeta)}$ is a $U_q(\mathfrak{sl}_2)$-submodule. Hence, we can assume $V=V_{(\zeta)}$ for some $\zeta\in\mathbb{k}$, i.e. $C-\zeta\cdot\mathrm{Id}$ acts nilpotently on V. In that case, clearly $\zeta=\zeta_i$ for all $1\leq i\leq m$. Hence, by Lemma 3(b), all subsequent quotients $\{V_i/V_{i-1}\}_{i=1}^m$ are isomorphic to some $L(n,\varepsilon)$. Thus:

- $\dim V=m\cdot\dim L(n,\varepsilon)=m\cdot(n+1)$
- $\dim V_\mu=m\cdot\dim L(n,\varepsilon)_\mu\ \forall\mu$ (where V_μ denotes K-weight spaces)

In particular, $\dim V_{\varepsilon q^n}=m$, $\dim V_{q^2\varepsilon q^n}=0$. Choose a basis $u_1,\ldots,u_m$ of V_λ, $\lambda:=\varepsilon q^n$. First, we claim that they generate all V as $U_q(\mathfrak{sl}_2)$-module. (If not, then the quotient $V/\underbrace{U_q(\mathfrak{sl}_2)(\mathrm{Span}\{u_1,\ldots,u_m\})}_{V'}$ is a nonzero $U_q(\mathfrak{sl}_2)$-mod but using its Jordan-Hölder filtration we see that all subsequent quotients are $\simeq L(n,\varepsilon)$, which is not possible as $(V/V')_\lambda=0$. Contradiction!)

On the other hand, $\dim V=n\cdot(n+1)=\sum_{i=1}^m\dim(U_q(\mathfrak{sl}_2)u_i)$. Therefore, we actually have $V\simeq\bigoplus_{i=1}^m U_q(\mathfrak{sl}_2)u_i\simeq\bigoplus_{i=1}^m L(n,\varepsilon)$. ∎

As we see, the proof is crucially based on the quantum Casimir C. In fact, we shall prove next time that C generates the center if $q\neq\sqrt[\cdot]{1}$.

> **Theorem 2**: If $q\neq\sqrt[\cdot]{1}$, then the center of $U_q(\mathfrak{sl}_2)$ is a polynomial algebra in C, i.e. $Z(U_q(\mathfrak{sl}_2))\simeq\mathbb{k}[C]$